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n-Groupoids from n-truncated simplicial objects as a solution to a universal problem

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Abstract

Forcing, in a certain fibration $\tau:\mathbb{E}\to\mathbb{B}$, a terminal object to be also initial, i.e. imposing some coherent splittings on some maps, allows one to recover n-groupoids from n-truncated simplicial objects and to control the machinery of the nerve functor for n-groupoids. © 2000 Elsevier Science B.V. All rights reserved.

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It was shown in [6] that, given any left exact category $\mathbb{B}$, the categories $S\text{-}Simpl_1\,\mathbb{B}$, $S\text{-}Simpl_2\,\mathbb{B}$,..., $S\text{-}Simpl_n\,\mathbb{B}$ of split $1,2,\dots,n$-truncated simplicial objects in $\mathbb{B}$ could be obtained iteratively by using a unique generic construction, namely that of the category of morphisms split "modulo a monad (T,λ,μ)". More precisely, given any left exact fibration $\tau:\mathbb{E}\to\mathbb{B}$ and any left exact monad (T,λ,μ) on $\mathbb{E}$, we denote by $T\mathbb{E}$ the category of the T-elements in $\mathbb{E}$. This category has objects $\underline{X}_1$defined by the triples (X_0,d,l) consisting of an object X_0 in $\mathbb{E}$, a map $d:X_1\to TX_0$ in the fibre of τ and a map $l:X_0\to X_1$ in $\mathbb{E}$ such that $d.l=\lambda X_0$:

$$\begin{tikzcd} & X_1 \arrow[d, "d"] \\ X_0 \arrow[ur, "l"] \arrow[r, "\lambda X_0"'] & TX_0 \end{tikzcd}$$

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Such a T-element is thus, properly speaking, a map d in a fibre of τ which is split by l modulo the monad (T, λ, μ).

There is again a left exact fibration $\tau_0 : T\mathbb{E} \to \mathbb{E}$ defined by $\tau_0(\underline{X}_1) = X_0$ and a left exact monad (T_1, λ_1, μ_1) on $T\mathbb{E}$ with $T_1(\underline{X}_1) = (X_1, d_0, l_0)$, where d_0 is determined by the following pullback:

$$\begin{tikzcd} X_1 \arrow[d, "d"'] & & X_1\langle d\rangle \arrow[ll, "m\langle d\rangle"'] \arrow[d, "d_0"] \\ TX_0 & T^2X_0 \arrow[l, "\mu X_0"] & TX_1 \arrow[l, "Td"] \end{tikzcd}$$

and $l_0 = [1_{X_1}, \lambda X_1]$.

The map Tl is a section of $\mu X_0.Td$ and produces a section $l\langle d\rangle$ of $m\langle d\rangle$. Whence the natural transformation $\lambda_1\underline{X_1} = (l, l\langle d\rangle) : \underline{X_1} \to T_1\underline{X_1}$, with $\lambda_1 X_1$ cartesian.

Now $T_1^2\underline{X_1}$ is determined by the following pullback:

$$\begin{tikzcd} X_1\langle d\rangle \arrow[d, "d_0"'] & & X_1^2\langle d\rangle \arrow[ll, "m\langle d_0\rangle"'] \arrow[d, "d_0"] \\ TX_1 & T^2X_1 \arrow[l, "\mu X_1"] & TX_1\langle d\rangle \arrow[l, "Td_0"] \end{tikzcd}$$

Then the equality $\mu X_0.Td.\mu X_1.Td_0 = \mu X_0.\mu TX_0.T^2d.Td_0 = \mu X_0.T\mu X_0.T^2d.Td_0 = \mu X_0.Td.Tm\langle d\rangle$ produces a unique map $m_1\langle d\rangle : X_1^2\langle d\rangle \to X_1\langle d\rangle$ such that $d_0.m_1\langle d\rangle = Tm\langle d\rangle.d_0$ and $m_1\langle d\rangle.m\langle d\rangle = m\langle d\rangle.m\langle d_0\rangle$.

Whence the natural transformation $\mu_1\underline{X_1} = (m\langle d\rangle, m_1\langle d\rangle)$ with $\mu_1\underline{X_1}$ cartesian. To check that λ_1 and μ_1 satisfy the axioms of a monad is straightforward.

Therefore, this construction $T\mathbb{E}$ appears to be the beginning of an inductive process. Starting from the final fibration: $\mathbb{B} \to 1$ and the identity monad on $\mathbb{B}$, the first step $T_0\mathbb{B}$ of this construction determines the category $Pt\,\mathbb{B}$ of the split epimorphisms in $\mathbb{B}$ and the monad (T_0, λ_0, μ_0) of [4]. The second step $T_1\mathbb{B}$ is just the category $S\text{-}Simpl_1\,\mathbb{B}$ and the $(n+1)$th step the category $S\text{-}Simpl_n\,\mathbb{B}$, creating in this way the whole tower of fibrations:

$$Pt\,\mathbb{B} \leftarrow S\text{-}Simpl_1\,\mathbb{B} \leftarrow S\text{-}Simpl_2\,\mathbb{B} \ldots S\text{-}Simpl_{n-1}\,\mathbb{B} \leftarrow S\text{-}Simpl_n\,\mathbb{B} \ldots$$

All the categories $S\text{-}Simpl_n\,\mathbb{B}$ are embedded in $S\text{-}Simpl\,\mathbb{B}$ the category of the split simplicial objects in $\mathbb{B}$ by a strict morphism of monads between (T_n, λ_n, μ_n) and the monad (T, λ, μ) on $S\text{-}Simpl\,\mathbb{B}$ shifting the index by one [12]. There is a natural comparison functor between $Alg\, T_n = K_{n+1}\mathbb{B}$ and $Alg\, T = Simpl\, \mathbb{B}$, the category of the simplicial objects in $\mathbb{B}$, which is actually an embedding. In particular, $Alg\, T_0$ is the category $Grd\, \mathbb{B}$ of the internal groupoids in $\mathbb{B}$, see [4], and, at this level, the comparison functor is the classical Grothendieck nerve functor.

On the other hand, it was shown in [4] that (in the same way as the category $Grd\, \mathbb{B}$ is monadic (owing to the monad (T_0, λ_0, μ_0)) above $Pt\, \mathbb{B}$), the category 2-$Grd\, \mathbb{B}$ of the internal 2-groupoids in $\mathbb{B}$ is monadic above the category N-$Grd\, \mathbb{B}$ of the normalized groupoids in $\mathbb{B}$, i.e. groupoids which are internally equipped with the choice of a point in every connected component and a map between this point and any other point in this component. More generally, the notion of normalized n-groupoid was brought out in [5] and was used in the proof of the monadicity of the category n-$Grd\, \mathbb{B}$ of the internal n-groupoids in $\mathbb{B}$ above the category N-$(n-1)$-$Grd\, \mathbb{B}$ of the normalized $(n-1)$-groupoids in $\mathbb{B}$, producing a new tower of fibrations:

$$Pt\, \mathbb{B} \leftarrow N\text{-}Grd\, \mathbb{B} \leftarrow N\text{-}2\text{-}Grd\, \mathbb{B} \ldots N\text{-}(n-1)\text{-}Grd\, \mathbb{B} \leftarrow N\text{-}n\text{-}Grd\, \mathbb{B} \ldots$$

This work is devoted to the study of the relation between the two previously mentioned towers of fibrations. This relation is actually based upon a very simple observation. Given a left exact fibration $\tau : \mathbb{E} \to \mathbb{B}$, with a terminal object in each fibre and given a monad (T, λ, μ) on $\mathbb{E}$, it is always possible to associate to the fibration τ, a fibration $\hat{\tau} : \hat{\mathbb{E}} \to \mathbb{B}$ in which the terminal objects in the fibres are also initial, with respect to the monad (T, λ, μ). If we denote by $\check{T}\mathbb{E}$ the result of the combination of the two constructions: first $\tau_0 : T\mathbb{E} \to \mathbb{E}$, from $\tau : \mathbb{E} \to \mathbb{B}$, then $\widehat{\tau_0} : \widehat{T\mathbb{E}} = \check{T}\mathbb{E} \to \mathbb{E}$, we get a new inductive process, which starting from the final fibration: $\mathbb{B} \to 1$ and the identity monad produces the tower of the normalized n-groupoids in $\mathbb{B}$.

Thus, the construction which forces the terminal object in a fibre to be also initial determines the normalized n-groupoids from the split n-simplicial objects. There is hidden here, as intuitively felt in “Low-dimensional geometry of the notion of choice” [4], a kind of $1, 2, \ldots, n$-geometry of the notion of choice.

As a consequence, there is a natural comparison functor $W_n : \check{T}_n\mathbb{B} = N\text{-}n\text{-}Grd\, \mathbb{B} \to T_n\mathbb{B} = S\text{-}Simpl_n\, \mathbb{B}$ which is a strict morphism of monads and produces a comparison functor $v_{n+1} : Alg\, \check{T}_n = (n+1)\text{-}Grd\, \mathbb{B} \to Alg\, T_n = K_{n+1}\mathbb{B}$ which is the nerve functor for the n-groupoids in $\mathbb{B}$. Whence, in the case of the n-groupoids, we have a synthetic description of the nerve which is opposite and complementary to the geometrico-combinatorial way of the “pasting” description in the case of n-categories [1,10,14,16–19].

One other consequence is some new light on the characterization of nerves. A student of Brown, Dakin, introduced in his thesis [10] the notion of T-complex which consists of a pair (X, T) of a simplicial set X and a graded subset $T = (T_i)_{i \geq 1}$ of X with $T_i \subset X_i$. The elements of T are called thin, and the three following axioms must be satisfied:

(A_1) All degenerate elements of X are thin.
(A_2) Any horn in X has a unique thin filler.
(A_3) If all but one of the faces of a thin element of X are themselves thin, then so also is the remaining face thin.

Another student of his, Ashley, proved the equivalence of the category of T-complexes and the category of crossed complexes [2], what is a non-obelian version of the Dold–Kan theorem. Modulo the result of Brown and Higgins concerning the equivalence of the category of crossed complexes and that of the ∞-groupoids in sets [9] (among their remarkable series of equivalences [7,8], see also [15]) it was clear that the category of T-complexes was equivalent to the category of ∞-groupoids and that consequently the notion of T-complex characterized the nerves of the ∞-groupoids in the set theoritical context.

Unfortunately, the notion of T-complex, because of axiom (A_2) and the large number of possible horns involved in it, does not fit internal context. Let us call Θ-complex the same data as a T-complex but satisfying only:
(B_1) All degenerate elements of X are thin.
(B_2) Any *initial* horn (see below) in X has a unique thin filler.
(B_3) If all but the *last one* of the faces of a thin element in X are themselves thin, then so also is the last face thin.

By an initial horn, we mean a simplex whose *last* (and not any one) face is missing. This much more economical notion makes sense in any internal context since (B_2) implies that T_i is actually isomorphic to a given simplicial kernel of X_i and consequently the category Θ-$\mathbb{B}$ of internal Θ-complexes in $\mathbb{B}$ can be easily defined (see Section 5).

More precisely, it is shown here that the category n-*Grd* $\mathbb{B}$ of n-groupoids in $\mathbb{B}$ is equivalent to Θ_n-$\mathbb{B}$, the full subcategory of Θ-$\mathbb{B}$ of Θ-complexes of rank n ($T_m = X_m\ \forall m > n$) and consequently that a Θ_n-complex gives a simplicial presentation of a n-groupoid. When $\mathbb{E}$ is additive, any simplicial object has a unique canonical structure of Θ-complex. The general method, here, based upon an inductive algebraicization of the definition of the nerve, will leave aside the geometric description in order to highlight algebraic property: the nerve functor v_{n+1} is cofibrant on the hypomorphisms (i.e. the maps in the fibres).

This paper is organized along the following line:
(1) The associated pointed fibration with respect to a monad: the construction $\hat{\tau}$ is given.
(2) The case of the T-elements of $\mathbb{E}$: the categories $T\mathbb{E}$ and $\check{T}\mathbb{E}$ are described.
(3) The tower of the normalized n-groupoids in $\mathbb{E}$.
(4) The nerve functor for n-groupoids.
(5) Characterization of the nerve of n-groupoids: introduction of the notion of Θ_n-complex.
(6) The nerve functor is cofibrant on the hypomorphisms.
(7) Normalized monads: a tool which allows characterization of the class of algebras.
(8) Last step: end of the inductive proof.

1. The associated pointed fibration with respect to a monad

We shall work in the following context. Let $\tau : \mathbb{E} \to \mathbb{B}$ be a left exact fibration, i.e. with each fibre left exact and each change of base functor left exact. The choice of a terminal object kZ in each fibre above an object Z of $\mathbb{B}$ determines a right adjoint right inverse to τ. We shall denote by $\omega : 1 \to k\tau$ the associated projection. Then a map $f : X \to Y$ in $\mathbb{E}$ is cartesian if and only if the following square is a pullback:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\omega X"'] & Y \arrow[d, "\omega Y"] \\ k\tau X \arrow[r, "k\tau f"'] & k\tau Y \end{tikzcd}$$

Let us call hypomorphism any map in $\mathbb{E}$ whose image by τ is an identity map, that is any map in a fibre. Then the class of hypomorphisms is pullback amenable in $\mathbb{E}$, which means that there is always a pullback of an hypomorphism along any map which is a hypomorphism (but a pullback of a hypomorphism is not always a hypomorphism, of course). From now on, we shall always choose by construction an identity map as a pullback of an identity map.

When furthermore the category $\mathbb{B}$ is left exact (which we shall suppose now on), the category $\mathbb{E}$ is again left exact and τ is left exact as a functor.

Main example. Let $\mathbb{B}$ be a left exact category and $S\text{-}Simpl_n\,\mathbb{B}$ denote the category of the split n-simplicial objects in $\mathbb{B}$ (i.e. split simplicial objects truncated at level n, where a simplicial object is split when there is an extra family of maps, namely the splittings $s_{n+1} : X_n \to X_{n+1}$ satisfying the classical equations at their own level). Then the functor $\tau_{n-1} : S\text{-}Simpl_n\,\mathbb{B} \to S\text{-}Simpl_{n-1}\,\mathbb{B}$ which erases the last level is a left exact fibration.

Definition 1.1. The left exact fibration τ is pointed if in each fibre the terminal object is also initial.

Example 1.1. If we denote by $Pt\,\mathbb{B}$ the category whose objects are the split epimorphisms in $\mathbb{B}$ and morphisms the commutative squares between such data, then the functor $p : Pt\,\mathbb{B} \to \mathbb{B}$ which associates its codomain to any split epimorphism is a pointed fibration. This category $Pt\,\mathbb{B}$ should be thought of as the category of split simplicial objects truncated at level 0.

We shall suppose now we have also a monad (T, λ, μ) on $\mathbb{E}$.

Definition 1.2. (1) The monad is connected to τ when the image by T of a terminal object in a fibre is again a terminal object in a fibre (in general not the same one).

(2) The monad is transverse to τ when both the natural transformations λ and μ are cartesian, i.e. λX and μX are always cartesian maps, for any object X in $\mathbb{E}$.

Example 1.2. The category $\mathbb{B}$ being left exact, it is well known [11] that the category $S\text{-}Simpl_n\,\mathbb{B}$ is embedded into the category $S\text{-}Simpl\,\mathbb{B}$ of the split simplicial objects in $\mathbb{B}$ by means of iterated simplicial kernels. Then the monad (T,λ,μ) on $S\text{-}Simpl\,\mathbb{B}$ determined by shifting of the index and whose category of algebras is $Simpl\,\mathbb{B}$, the category of simplicial objects in $\mathbb{B}$, is actually stable on $S\text{-}Simpl_n\,\mathbb{B}$ as a monad (T_n,λ_n,μ_n) which is connected and transverse to the left exact fibration τ_{n-1} (see [6, Propositions 14] and [2, Corollary 1]).

The monad (T_0,λ_0,μ_0) on $Pt\,\mathbb{B}$, defined on the same model as (T_n,λ_n,μ_n), is also connected and transverse to the pointed fibration p. Here, the category of algebras $Alg\,T_0$ is nothing but the category $Grd\,\mathbb{B}$ of internal groupoids in $\mathbb{B}$ [4], which will be very important with respect to our project.

Remark 1.1. We shall suppose from now on that the isomorphism: $\omega TkZ : TkZ \to k\tau TkZ$ is actually an identity map for any object Z in $\mathbb{B}$, without any loss of generality. This is actually satisfied by our main examples. This implies that $k\tau Tk = Tk$ and that, for every object X in $\mathbb{E}$, we have $k\tau T\omega X.\omega TX = T\omega X$.

Remark 1.2. When the functor T is left exact, and the monad (T,λ,μ) is connected to the left exact fibration τ, then clearly the functor T preserves the cartesian maps, owing to the previous equality.

Proposition 1.1. *When a monad (T,λ,μ) on $\mathbb{E}$ is connected to a left exact fibration τ, then it determines a canonical monad (T_B,λ_B,μ_B) on $\mathbb{B}$ and makes $k:\mathbb{B}\to\mathbb{E}$ a strict morphism of monads $T_B \to T$.*

Proof. Let us set $T_BZ = \tau TkZ$, $\lambda_BZ = \tau\lambda kZ$, $\mu_BZ = \tau\mu kZ$. Whence $kT_B = k\tau Tk = Tk$, $k\lambda_B = \lambda k$ and $k\mu_B = \mu k$. □

Remark 1.3. The fact that k is a strict morphism of monads determines an extension functor $Alg\,k : Alg\,T_B \to Alg\,T$ by $Alg\,k(Z,z) = (kZ,kz)$.

Remark 1.4. In general there is no such extension of $\tau:\mathbb{E}\to\mathbb{B}$ at the level of the categories of algebras. But when τ is pointed, the initial map σX in the fibre: $k\tau X \to X$ determines a lax morphism of monads through the map $\tau T\sigma : T_B\tau = \tau Tk\tau \to \tau T$ which allows in this case an extension functor $Alg\,\tau : Alg\,T \to Alg\,T_B$ defined by

$$Alg\,\tau(X,x) = (\tau X, \tau x.\tau T\sigma X).$$

Proposition 1.2. *If (T,λ,μ) is connected to the pointed fibration τ, then $Alg\,k$ is a left adjoint right inverse to $Alg\,\tau$.*

Proof. Straightforward. □

We now consider a left exact fibration $\tau : \mathbb{E} \to \mathbb{B}$ and a left exact monad (T, λ, μ) connected and transverse to τ, and describe its associated universal pointed fibration $\hat{\tau} : \hat{\mathbb{E}} \to \mathbb{B}$ with respect to the monad (T, λ, μ). So let us denote by $\hat{\mathbb{E}}$ the category whose objects are the pairs (X, s) of an object X in $\mathbb{E}$ and a splitting s of the terminal map $\omega TX : TX \to k\tau TX$ in the fibre of TX satisfying $(*)$: $\mu X.Ts = s.k\tau(\mu X.Ts)$ and morphisms between (X, s) and (X', s') the maps $f : X \to X'$ such that $Tf.s = s'.k\tau Tf$. Clearly equality $(*)$ is equivalent to $s.\omega TX.\mu X.Ts = \mu X.Ts$

Remark 1.5. There is an obvious forgetful functor $W : \hat{\mathbb{E}} \to \mathbb{E}$ defined by $W(X, s) = X$. Since the functor T is left exact, W is left exact. In fact $\hat{\tau} = \tau.W : \hat{\mathbb{E}} \to \mathbb{B}$ is actually a fibration.

Remark 1.6. This $\hat{\tau}$ is clearly left exact as a fibration since each fibre of $\hat{\tau}$ is stable by pullback and $\hat{k}Z = (kZ, 1_{TkZ})$ is a terminal object in the fibre of Z. We shall denote by $\hat{\omega}(X, s) : (X, s) \to (k\tau X, 1) = \hat{k}\hat{\tau}(X, s)$ the map in $\hat{\mathbb{E}}$ determined by $\omega X : X \to k\tau X$. Since $T\omega X = k\tau T\omega X.\omega TX$, we have $T\omega X.s = k\tau T\omega X = 1.k\tau T\omega X$.

Furthermore we have clearly $W.\hat{k} = k$.

Proposition 1.3. *The functor W preserves and reflects the cartesian maps and the hypomorphisms. It reflects the identity maps and is discretely fibrant on the cartesian maps.*

Proof. Preservation comes from the fact that W is left exact and that $\tau.W = \hat{\tau}$. Reflection is straightforward. The last assertion means that given an object (X', s') in $\hat{\mathbb{E}}$ and $f : X \to X'$ in $\mathbb{E}$ a cartesian map, there exists a unique cartesian map in $\hat{\mathbb{E}}$ above f. □

Remark 1.7. Since μX is cartesian the splitting s of ωTX determines a splitting $\bar{s}$ of ωT^2X such that $\mu X.\bar{s} = s.k\tau\mu X$. Hence $\lambda TX.s = \bar{s}.k\tau\lambda TX$ and $T\lambda X.s = \bar{s}.k\tau T\lambda X$.

Proposition 1.4. *The map s being a splitting of ωTX, the equality $\mu X.Ts = s.k\tau(\mu X.Ts)$ holds if and only if $Ts = \bar{s}.k\tau Ts$ or equivalently $\bar{s}.\omega T^2X.Ts = Ts$.*

Proof. The images by τ of the two maps Ts and $s.k\tau Ts$ being equal, these two maps are equal if and only if their compositions with the cartesian map μX are equal. But $\mu X.\bar{s}.k\tau Ts = s.k\tau\mu X.k\tau Ts = s.k\tau(\mu X.Ts)$, and $k\tau Ts = \bar{s}.\omega T^2X$ by the naturality of ω. □

Theorem 1.5. *The fibration $\hat{\tau}$ is pointed. The monad (T, λ, μ) on $\mathbb{E}$ extends to a monad $(\hat{T}, \hat{\lambda}, \hat{\mu})$ on $\hat{\mathbb{E}}$ which is left exact, connected and transverse to $\hat{\tau}$. Furthermore the functor W is a strict morphism of monads.*

Proof. (1) The map λX being cartesian, the splitting s of ωTX produces a splitting σ of ωX such that $\lambda X.\sigma = s.k\tau\lambda X$. Let us prove that σ determines a morphism $\hat{\sigma}(X, s)$:

$\hat{k}\hat{\tau}(X,s) \to (X,s)$ in $\hat{\mathbb{E}}$, i.e. that $T\sigma = s.k\tau T\sigma$. But these two maps have the same image by τ, so it is sufficient to prove their equality by composition with the cartesian map $T\lambda X$:

$$\begin{aligned} T\lambda X.s.k\tau T\sigma &= \bar{s}.k\tau T\lambda X.k\tau T\sigma = \bar{s}.k\tau T(\lambda X.\sigma) = \bar{s}.k\tau T(s.k\tau\lambda X) \\ &= \bar{s}.k\tau Ts.Tk\tau\lambda X = Ts.Tk\tau\lambda X = T\lambda X.T\sigma. \end{aligned}$$

Thus $\hat{\sigma}(X,s)$ determines a natural section of $\hat{\omega}(X,s)$ and makes $\hat{\tau}$ a pointed fibration.

(2) We can set $\hat{T}(X,s) = (Tx, \bar{s})$ since the maps $\mu TX.T\bar{s}$ and $\bar{s}.k\tau(\mu TX.T\bar{s})$ have the same image by τ. Since the map μX is cartesian, we have $\mu X.\bar{s}.k\tau(\mu TX.T\bar{s}) = s.k\tau\mu X.k\tau(\mu TX.T\bar{s}) = s.k\tau\mu X.k\tau(T\mu X.T\bar{s}) = s.k\tau\mu X.k\tau(Ts.Tk\tau\mu X) = \mu X.Ts.Tk\tau\mu X = \mu X.T\mu X.T\bar{s} = \mu X.\mu TX.T\bar{s}$.

This functor $\hat{T}$ is clearly left exact and satisfies $W.\hat{T} = T.W$. We saw that $T\lambda X.s = \bar{s}.k\tau T\lambda X$ whence there is a map $\hat{\lambda}(X,s) : (X,s) \to \hat{T}(X,s)$ given by λX. On the other hand, the maps $T\mu X.\bar{s}^2$ and $s.k\tau T\mu X$ have the same image by τ and $\mu X.T\mu X.\bar{s}^2 = \mu X.\mu TX.\bar{s}^2 = \mu X.\bar{s}.k\tau\mu TX$. Thus there is a map $\hat{\mu}(X,s) : \hat{T}^2(X,s) \to \hat{T}(X,s)$ given by μX, and W is a strict morphism of monads.

(3) $(\hat{T}, \hat{\lambda}, \hat{\mu})$ is connected and transverse to $\hat{\tau}$ since on the one hand $W.\hat{k} = k$ and W reflects the identity maps, and on the other hand it reflects the cartesian maps. □

Example 1.3. When τ is pointed, with $\sigma : k\tau \to 1$ the natural section of ω, we have always $\sigma TX.k\tau T\sigma X = T\sigma X$. Now σTX is the only section of ωTX, and $\overline{\sigma TX} = \sigma T^2 X$. The equation $\sigma T^2 X.k\tau\sigma TX = T\sigma TX$, according to Proposition 1.4 means that $(X, \sigma TX)$ belongs to $\hat{\mathbb{E}}$ and defines an inverse to W. Therefore when τ is pointed, we have $\hat{\mathbb{E}} \simeq \mathbb{E}$. In particular we have $\widehat{Pt\,\mathbb{B}} \simeq Pt\,\mathbb{B}$.

Example 1.4. The generic example. We shall consider now the fibration $\tau_0 : S\text{-}Simpl_1\mathbb{B} \to Pt\,\mathbb{B}$, where $\mathbb{B}$ is left exact. An object of $S\text{-}Simpl_1\,\mathbb{B}$ is just a reflexive graph in $\mathbb{B}$:

$$\begin{tikzcd} X_0 \arrow[r] & X_1 \arrow[l, bend right, "d_0"'] \arrow[l, bend left, "d_1", "s_0"'] \end{tikzcd}$$

with a section $s_1 : X_0 \to X_1$ such that:

(1) $d_1 s_1 = 1_{X_0}$, (2) $d_0.s_1.d_0 = d_0.s_1.d_1$, (3) $s_0.d_0.s_1 = s_1.d_0.s_1$.

The functor τ_0 associates to this graph the splitting of the idempotent map $d_0.s_1$.

A normalized groupoid in $\mathbb{B}$ is just a groupoid in $\mathbb{B}$ with a splitting (the normalization) of its underlying reflexive graph. A morphism between normalized groupoids is just an internal functor which preserves the normalization. We denote by $N\text{-}Grd\,\mathbb{B}$ the category of the normalized groupoids in $\mathbb{B}$ and by $W : N\text{-}Grd\,\mathbb{B} \to S\text{-}Simpl_1\mathbb{B}$ the obvious forgetful functor.

Proposition 1.6. *The category* $\widehat{S\text{-}Simpl_1}\,\mathbb{B}$ *and* $N\text{-}Grd\,\mathbb{B}$ *are isomorphic.*

Proof. The functor (T_1, λ_1, μ_1) on $S\text{-}Simpl_1\,\mathbb{B}$ is defined in the following way: given a split reflexive graph $(\underline{X_1}, s_1)$ then $T_1(\underline{X_1}, s_1)$ is the following reflexive graph:

$$\begin{tikzcd}
X_1 \arrow[r, shift left=4.5] & X_1[d_0, d_1] \arrow[l, shift right=9, "p_0"'] \arrow[l, shift left=1.5, "s_0"'] \arrow[from=1-1, r, shift right=6, "p_1", "s_1"']
\end{tikzcd}$$

where $X_1[d_0, d_1]$ is the simplicial kernel of the pair (d_0, d_1) and p_0, p_1 the two first canonical propections. In the set theoretical context, this graph $T_1(\underline{X_1}, s_1)$ has the maps of X_1 as objects and the triangle of maps as morphisms. Now an object of $S\text{-}\widehat{Simpl_1}\,\mathbb{B}$ is just a split reflexive graph $(\underline{X_1}, s_1)$ endowed with a section of $\omega T_1(\underline{X}_1, s_1)$ which is given by the following diagram:

$$\begin{tikzcd}
X_1 \arrow[r] \arrow[d, "1_{X_1}"'] & X_1[d_0, d_1] \arrow[l, shift right=3, "p_0"'] \arrow[l, shift left=3, "p_1"] \arrow[d, "{[p_0, p_1]}"] \\
X_1 \arrow[r] & X_1[d_0] \arrow[l, shift right=3, "p_0"'] \arrow[l, shift left=3, "p_1"]
\end{tikzcd}$$

where the lower pair is the kernel pair of d_0. Thus this section only consists of a section φ of $[p_0, p_1]$ such that $p_i.\varphi = p_i$, $\varphi.s_i = s_i$, $0 \le i \le 1$. Now the map $p_2.\varphi : X_1[d_0] \to X_1$ completes the structure of an internal groupoid in $\mathbb{E}$ (as an object of $Alg\,T_0$), which is normalized by s_1.

Conversely, given $\underline{X_1}$ a groupoid with a normalization s_1 then the factorization $[d_0, d_1, d_2] : X_1[d_0] \to X_1[d_0, d_1]$ produces the desired section of $\omega T_1(X_1, s_1)$. □

Remark 1.8. The monad $(\widehat{T_1}, \widehat{\lambda_1}, \widehat{\mu_1})$ induced on $N\text{-}Grd\,\mathbb{B}$ is nothing but the monad (T_1, λ_1, μ_1) of the non-commutative triangles described in [3], where it is shown that the category $Alg\,T_1$ is the category $2\text{-}Grd\,\mathbb{B}$ of the internal 2-groupoids in $\mathbb{B}$. The functor W being a strict morphism of monads, it produces an extention at the level of the categories of algebras: $v_2 : 2\text{-}Grd\,\mathbb{B} \to K_2\mathbb{B}$ where $K_2\mathbb{B}$ is the category of algebras of the monad (T_1, λ_1, μ_1) on $S\text{-}Simpl_1\,\mathbb{B}$ and therefore is a sub-category of $Simpl\,\mathbb{B}$ since (T_1, λ_1, μ_1) is a restriction of the monad (T, λ, μ) on $S\text{-}Simpl\,\mathbb{B}$ whose category of algebras is $Simpl\,\mathbb{B}$.

In [4], this functor v_2 was shown to be the nerve functor for 2-groupoids. Therefore, this is the construction of the associated pointed fibration to $\tau_0 : S\text{-}Simpl_1\,\mathbb{B} \to Pt\,\mathbb{B}$ which determines completely the nerve functor for the 2-groupoids. The aim of this paper is to study how this same construction is related to the nerve functor for the n-groupoids [6]: $v_n : n\text{-}Grd\,\mathbb{B} \to K_n\mathbb{B}$.

The construction $\hat{\tau}$ has a universal property:

Proposition 1.7. *Given a pointed fibration $\tau' : \mathbb{E}' \to \mathbb{B}'$, a monad (T', λ', μ') on $\mathbb{E}'$, connected to τ', and a commutative square such that G is a strict morphism of monads*:

$$\begin{tikzcd}
\mathbb{E}' \arrow[r, "G"] \arrow[d, shift right, "\tau'"'] & \mathbb{E} \arrow[d, shift right, "\tau'"'] \\
\mathbb{B}' \arrow[r, "F"] \arrow[u, shift right, "k'"'] & \mathbb{B} \arrow[u, shift right, "k'"']
\end{tikzcd}$$

there is a unique factorization $\hat{G} : \mathbb{E}' \to \hat{\mathbb{E}}$ such that *$W.\hat{G} = G$, $\hat{\tau}.\hat{G} = F.\tau'$, and $\hat{G}$ is a strict morphism of monads.*

Proof. Merely set $\hat{G}(X') = (GX', G(\sigma' T' X'))$. $\square$

2. The case of the T-elements of $\mathbb{E}$

Let us recall from [6] the following definitions and results. We shall suppose here that τ is a left exact fibration and (T, λ, μ) a monad on $\mathbb{E}$, no relation between them being assumed.

Definition 2.1. A T-element in $\mathbb{E}$ is a triple $\underline{X_1} = (X_0, d, l)$ with X_0 an object in $\mathbb{E}$, $d : X_1 \to TX_0$ a hypomorphism and $l : X_0 \to X_1$ a map satisfying $d.l = \lambda X_0$. A T-morphism between $\underline{X_1}$ and $\underline{X_1'}$ is a pair (f_0, f_1), $f_0 : X_0 \to X_0'$ and $f_1 : X_1 \to X_1'$ of morphisms in $\mathbb{E}$ such that $d'.f_1 = Tf_0.d$ and $f_1.l = l'.f_0$.

Thus, a T-element is nothing but a hypomorphism which is split "modulo" the monad (T, λ, μ). We shall denote by $T\mathbb{E}$ the category of the T-elements in $\mathbb{E}$. There is a forgetful functor $\tau_0 : T\mathbb{E} \to \mathbb{E}$ associating X_0 to $\underline{X_1}$, which is obviously a fibration, a T-morphism (f_0, f_1) being cartesian if and only if the following square is a pullback:

$$\begin{tikzcd}
X_1 \arrow[r, "f_1"] \arrow[d, "d"'] & X_1' \arrow[d, "d'"] \\
TX_0 \arrow[r, "Tf_0"] & TX_0'
\end{tikzcd}$$

Furthermore, τ_0 is left exact as a fibration and $k_1(X) = (X, 1_{TX}, \lambda X)$ is a terminal object in the fibre above the object X in $\mathbb{E}$.

There is on $T\mathbb{E}$ a monad (T_1, λ_1, μ_1) defined in the following way: given (X_1, d, l) a T-element, let us consider the following amenable pullback:

$$\begin{tikzcd}
X_1 \arrow[d, "d"'] & & X_1\langle d\rangle \arrow[ll, "m\langle d\rangle"'] \arrow[d, "d_0"] \\
TX_0 & T^2X_0 \arrow[l, "\mu X_0"] & TX_1 \arrow[l, "Td"]
\end{tikzcd}$$

We have $\mu X_0.Td.\lambda X_1 = \mu X_0.\lambda TX_0.d = d$, whence there is a unique map $l_0 : X_1 \to X_1\langle d\rangle$ satisfying $d_0.l_0 = \lambda X_1$ and $m\langle d\rangle.l_0 = 1$. Let us set $T_1\underline{X}_1 = (X_1, d_0, l_0)$. It is clearly an endofunctor on $T\mathbb{E}$ such that $\tau T_1\underline{X}_1 = X_1$.

On the other hand, the map Tl is a section of $\mu X_0.Td$ and produces a section $l\langle d\rangle$ of $m\langle d\rangle$, and a natural map $\lambda_1\underline{X_1} = (l, l\langle d\rangle) : \underline{X_1} \to T_1\underline{X_1}$ with $\lambda_1\underline{X}_1$ cartesian.

Now $T_1^2\underline{X_1}$ is determined by the following pullback:

$$\begin{tikzcd}
X\langle d\rangle \arrow[d, "d_0"'] & & X_1^2\langle d\rangle \arrow[ll, "m\langle d_0\rangle"'] \arrow[d, "d_0"] \\
TX_1 & T^2X_1 \arrow[l, "\mu X_1"] & TX_1\langle d\rangle \arrow[l, "Td_0"]
\end{tikzcd}$$

Then the equality $\mu X_0.Td.\mu X_1.Td_0 = \mu X_0.\mu TX_0.T^2d.Td_0 = \mu X_0.T\mu X_0.T^2d.Td_0 = \mu X_0.Td.Tm\langle d\rangle$ produces a unique map $m_1\langle d\rangle : X_1^2\langle d\rangle \to X_1\langle d\rangle$ such that $d_0.m_1\langle d\rangle = Tm\langle d\rangle.d_0$ and $m\langle d\rangle.m_1\langle d\rangle = m\langle d\rangle.m\langle d_0\rangle$. Thus there is a natural map $\mu_1\underline{X_1} = (m\langle d\rangle, m_1\langle d\rangle)$ with $\mu_1\underline{X_1}$ cartesian. To check that the natural transformations λ_1 and μ_1 satify the axioms of a monad is straightforward. Furthermore $T_1k_1(X) = (TX, 1_{T^2X}, \lambda TX) = k_1\tau_0T_1k_1(X)$, and thus:

Proposition 2.1. *The monad (T_1, λ_1, μ_1) is connected and transverse to τ_0. When T is left exact, then T_1 is left exact.*

Example 2.1. (1) When $\tau : \mathbb{B} \to 1$ is the terminal fibration, and (T, λ, μ) the identity monad *Id*, then *Id*-$\mathbb{B}$ is nothing but $Pt\,\mathbb{B}$ and the monad on it is just the monad (T_0, λ_0, μ_0).

(2) The construction $T\mathbb{E}$ is clearly the beginning of an inductive process, since, starting from a left exact fibration τ and a monad (T, λ, μ) on its domain $\mathbb{E}$, we constructed a left exact fibration τ_0 with a monad (T_1, λ_1, μ_1) on its domain $T\mathbb{E}$.

(3) One of the main results in [6] is that if τ is $\tau_{n-1} : S\text{-}Simpl_n\,\mathbb{B} \to S\text{-}Simpl_{n-1}\,\mathbb{B}$ and the monad (T, λ, μ) is (T_n, λ_n, μ_n) then $T_n(S\text{-}Simpl_n\,\mathbb{B})$ is $S\text{-}Simpl_{n+1}\,\mathbb{B}$ and the associated monad is $(T_{n+1}, \lambda_{n+1}, \mu_{n+1})$.

Thus the inductive process applied to the terminal fibration $\mathbb{B} \to 1$ and the identity monad produces the tower of the truncated split simplicial objects

$$1 \leftarrow \mathbb{B} \leftarrow Pt\,\mathbb{B} \leftarrow S\text{-}Simpl_1\,\mathbb{B} \ldots S\text{-}Simpl_{n-1}\,\mathbb{B} \leftarrow S\text{-}Simpl_n\,\mathbb{B} \ldots.$$

and the category $S\text{-}Simpl_n\,\mathbb{B}$ appears to be the category of the iterated *Id*-elements of $\mathbb{B}$.

We are now going to study the pointed fibration associated to $\tau_0 : T\mathbb{E} \to \mathbb{E}$ with respect to the monad (T_1, λ_1, μ_1), when the basic monad (T, λ, μ) on $\mathbb{E}$ is left exact.

Let us denote by $Pt_\tau Alg\, T$ the full subcategory of $Pt\, Alg\, T$ (i.e the category whose objects are the split epimorphisms in $Alg\, T$) whose objects are the split epimorphisms with their underlying maps in $\mathbb{E}$ hypomorphic. Then the restriction of the fibration $p : Pt\, Alg\, T \to Alg\, T$ to $Pt_\tau Alg\, T$ is again a left exact fibration, since τ is left exact.

Theorem 2.2. *When the monad (T, λ, μ) is left exact then the pointed fibration $\hat{\tau}_0$ associated to τ_0 with respect to (T_1, λ_1, μ_1) is given by the following pullback*:

$$\begin{tikzcd} \hat{T}\mathbb{E} \arrow[r, "\tilde{F}"] \arrow[d, "\hat{\tau}_0"'] & Pt_\tau Alg\, T \arrow[d, "p"] \\ \mathbb{E} \arrow[r, "F"'] & Alg\, T \end{tikzcd}$$

Proof. Given an object $\underline{X}_1 = (X_0, d, l)$ in $T\mathbb{E}$, a section of $\omega T_1 \underline{X}_1$ is just given by a section s_0 of $d_0 : X_1\langle d\rangle \to TX_1$ such that $s_0.\lambda X_1 = l_0$. But the object $X_1\langle d\rangle$ being defined as a pullback, a section of d_0 is the same thing as a map $x : TX_1 \to X_1$ such that $d.x = \mu X_0.Td$. Furthermore $s_0.\lambda X_1 = l_0$ if and only if $x.\lambda X_1 = 1_{X_1}$.

Now the map $s = (1_{X_0}, s_0)$ in $T\mathbb{E}$ must satisfy $(*)$ $\mu_1 \underline{X}_1.T_1 s = s.k_1 \tau_0(\mu_1 \underline{X}_1.T_1 s)$ in order to produce an object in $\hat{T}\mathbb{E}$. But this equation is equivalent to the following ones:

- at the level 0: $m\langle d\rangle.s_0 = 1_{TX_1}.\mu\langle d\rangle.s_0$, which is trivial;
- at the level $1 : m_1\langle d\rangle.(s_0^2.Ts_0) = s_0.Tm\langle d\rangle.Ts_0$ where s_0^2 is the unique splitting of d_0^2 such that $m\langle d_0\rangle.s_0^2 = s_0.\mu X_1.Td_0$. But the equality at level 1 holds if and only if it holds by composition by d_0 (straightforward) and by $m\langle d\rangle$. But
  (α) $m\langle d\rangle.m_1\langle d\rangle.s_0^2.Ts_0 = m\langle d\rangle.m\langle d_0\rangle.s_0^2.Ts_0 = m\langle d\rangle.s_0.\mu X_1.Td_0.Ts_0 = x.\mu X_1$.
  (β) $m\langle d\rangle.s_0.Tm\langle d\rangle.Ts_0 = x.Tx$.

In other words, the identity $(*)$ holds if and only if $x.\mu X_1 = x.Tx$, and $(\underline{X_1}, s)$ is in $\hat{T}\mathbb{E}$ if and only if x is an algebra on X_1 for the monad (T, λ, μ), and d a hypomorphism of algebra $d : (X_1, x) \to (TX_0, \mu X_0) = F(X_0)$. Consequently the map $l : X_0 \to X_1$ produces a map $\sigma : (TX_0, \mu X_0) \to (X_1, x)$ in $Alg\, T$ and the equality $d.l = \lambda X_1$ is equivalent to $d.\sigma = 1_{TX_0}$. $\square$

Remark 2.1. It is shown in [5] (Proposition 10) that since F has a right adjoint U, the functor $\tilde{F}$ has a right adjoint $\tilde{U}$ which associates to each split hypomorphism (d, s) of algebras, $d : (X, x) \to (Y, y)$, the following amenable pullback in $Alg\, T$:

$$\begin{tikzcd} X \arrow[d, shift right, "d"'] & U \arrow[l] \arrow[d, shift right, "\delta"'] \\ Y \arrow[u, shift right, "s"'] & TY \arrow[l, "y"] \arrow[u, shift right] \end{tikzcd}$$

On the other hand, the monad (T_0, λ_0, μ_0) defined on $Pt\,Alg\,T$ is actually stable on $Pt_\tau Alg\,T$.

Proposition 2.3. *The monad $(\widehat{T_1}, \widehat{\lambda_1}, \widehat{\mu_1})$ on $\hat{T}\mathbb{E}$ is given by the composition of the monad* (T_0, λ_0, μ_0) on $Pt_\tau Alg\,T$ *with the adjoint pair* $(\tilde{U}, \tilde{F})$.

Proof. Starting from (d,s), $d : (X_1, x) \to (TX_0, \mu X_0)$ an object in $\hat{T}\mathbb{E}$, then $\tilde{F}(d,s)$ is the same pair in $Alg\,T$, and $T_0\tilde{F}(d,s)$ is determined by the following pullback in $Alg\,T$:

$$\begin{tikzcd}
X_1 \arrow[d, shift right, "d"'] & X_1[d_0] \arrow[l, "p_1"'] \arrow[d, shift right, "p_0"'] \\
TX_0 \arrow[u, shift right, "s"'] & X_1 \arrow[l, "d"] \arrow[u, shift right, "s_0"']
\end{tikzcd}$$

Now $\tilde{U}T_0\tilde{F}(d,s)$ is the following pullback:

$$\begin{tikzcd}
X_1[d_0] \arrow[d, shift right, "p_0"'] & U \arrow[l] \arrow[d, shift right] \\
X_1 \arrow[u, shift right, "s_0"'] & TX_1 \arrow[l, "x"] \arrow[u, shift right]
\end{tikzcd}$$

but $d.x = \mu X_0.Td$ and $\tilde{U}T_0\tilde{F}(d,s)$ is (up to isomorphism) $\hat{T}_1(d,s)$. $\square$

Remark 2.2. It is clear that the category of algebras $Alg_\tau\, T_0$ of the monad (T_0, λ_0, μ_0) on $Pt_\tau Alg\,T$ is $Grd_\tau Alg\,T$ the category of the internal groupoids in $Alg\,T$ which are in the fibres of τ.

Following the previous proposition, the composition of the adjoint pairs (U_0, F_0) and $(\tilde{U}, \tilde{F})$:

$$Grd_\tau Alg\,T \xrightarrow{U_0} Pt_\tau Alg\,T \xrightarrow{\tilde{U}} \hat{T}\mathbb{E}$$

produces a comparison functor $\varphi : Grd_\tau Alg\,T \to Alg\,\hat{T}_1$, which is, in a way, the heart of the nerve functor for the n-groupoids, as shown in [6].

3. The tower of the normalized n-groupoids

When the left exact fibration τ is pointed and the monad (T, λ, μ) is connected to τ, we saw that $\tau : \mathbb{E} \to \mathbb{B}$ extends to a functor $Alg\,\tau : Alg\,T \to Alg\,T_B$ and therefore the objects of $Pt_\tau\, Alg\,T$ are actually the split epimorphisms which are hypomorphic not only with respect to τ but with respect to $Alg\,\tau$. Furthermore, we can associate to it the pointed fibration $\hat{\tau}_0 : \hat{T}\mathbb{E} \to \mathbb{E}$, with the monad $(\hat{T}_1, \hat{\lambda}_1, \hat{\mu}_1)$ which is connected to $\hat{\tau}_0$. It is again the beginning of an inductive process.

We shall denote by $\check{\tau}_{n-1} : \check{T}_n\mathbb{E} \to \check{T}_{n-1}\mathbb{E}$ the pointed fibration and by $(\check{T}_n, \check{\lambda}_n, \check{\mu}_n)$ the monad on $\check{T}_n\mathbb{E}$ obtained at the nth level of this process. Clearly $\check{T}_1\mathbb{E}=\hat{T}\mathbb{E}$, $\check{T}_2\mathbb{E}=\hat{\hat{T}}_1(\hat{T}\mathbb{E})$ etc.

Taking the pointed fibration $p : Pt\,\mathbb{B} \to B$ with the monad (T_0, λ_0, μ_0) as starting point, we saw that $\check{T}_1(Pt\,\mathbb{B})=\widehat{S\text{-}Simpl}\,\mathbb{B}=N\text{-}Grd\,\mathbb{B}$ the category of internal normalized groupoids in $\mathbb{B}$. We recalled that the category of algebras of (T_1, λ_1, μ_1) on $N\text{-}Grd\,\mathbb{B}$ is exactly the category $2\text{-}Gd\,\mathbb{B}$ of internal 2-groupoids in $\mathbb{B}$. So according to Theorem 2.2, the category $\check{T}_2(Pt\,\mathbb{B})$ is given by the following pullback:

$$\begin{tikzcd}
\check{T}_2(Pt\,\mathbb{B}) \arrow[r] \arrow[d] & Pt_\tau(2\text{-}Grd\,\mathbb{B}) \arrow[d, "p"] \\
N\text{-}Grd\,\mathbb{B} \arrow[r, "F_1"'] & 2\text{-}Grd\,\mathbb{B}
\end{tikzcd}$$

with $\tau : 2\text{-}Grd\,\mathbb{B} \to Grd\,\mathbb{B}$ associating to each 2-groupoid X_2 its underlying groupoid X_1 of 1-morphisms. But according to [5], Definition 4, this pullback defines precisely the category $N\text{-}2\text{-}Grd\,\mathbb{B}$ of the normalized 2-groupoids in $\mathbb{B}$.

We are now going to show that the inductive process $\check{T}$ applied to $p : Pt\,\mathbb{B} \to \mathbb{B}$ produces exactly the tower of the normalized n-groupoids in $\mathbb{B}$, equipped with the monad (T_n, λ_n, μ_n) whose category of algebras $Alg\,T_n$ is the category $(n+1)\text{-}Grd\,\mathbb{B}$ of the internal $(n+1)$-groupoids in $\mathbb{B}$. Thus, on the one hand, the inductive construction of the T-elements from the fibration $p : Pt\,\mathbb{B} \to \mathbb{B}$ produces the tower of the split n-simplicial objects and on the other hand, the inductive construction $\check{T}$ of the associated pointed fibration produces the tower of the normalized n-groupoids in $\mathbb{B}$. And so the construction which forces the terminal object in a fibre to be also initial, determines the normalized n-groupoids from the split n-simplicial objects.

Given $\mathbb{B}$ a left exact category, let $Grd\,\mathbb{B}$ denote the category of internal groupoids in $\mathbb{B}$ and $(\ )_0 : Grd\,\mathbb{B} \to \mathbb{B}$ (which is the extension to the categories of algebras of the fibration $p : Pt\,\mathbb{B} \to \mathbb{B}$, Remark 1.4) denotes the functor associating to each internal groupoid $\underline{X_1}$ its object of objects X_0. It has a right adjoint right inverse g_1 where g_1X is the kernel groupoid of the terminal map $X \to 1$.

Now, suppose we have defined the category $k\text{-}Grd\,\mathbb{B}$ and the fibration $(\ )_{k-1} : k\text{-}Grd\,\mathbb{B} \to (k-1)\text{-}Grd\,\mathbb{B}$ with a right adjoint right inverse g_k, $\forall k,\ 1 \le k \le n$. Then the category $(n+1)\text{-}Grd\,\mathbb{B}$ is the full subcategory of $Grd(n\text{-}Grd\,\mathbb{B})$ whose objects are the groupoids $\underline{X}_{n+1}$ in the fibres of $(\ )_{n-1} : n\text{-}Grd\,\mathbb{B} \to (n-1)\text{-}Grd\,\mathbb{B}$.

$$\begin{tikzcd}[column sep=large]
\underline{X}_{n+1} : X_n \arrow[r] & X_{n+1} \arrow[l, shift right=3, "d_0"'] \arrow[l, shift left=3, "d_1"] & X_{n+1}[d_0] \arrow[l, shift right=3, "p_0"'] \arrow[l, "p_1"] \arrow[l, shift left=3, "d_2"]
\end{tikzcd}$$

We shall denote by $(\ )_n : (n+1)\text{-}Grd\,\mathbb{B} \to n\text{-}Grd\,\mathbb{B}$ the functor associating X_n to $\underline{X}_{n+1}$. This functor has a right adjoint right inverse g_{n+1} where $g_{n+1}(X_n)$ is the kernel groupoid of the terminal map $X_n \to g_n(X_{n-1})$ in the fibre of X_n.

We saw that a normalized groupoid is a groupoid whose underlying reflexive graph is split. Let us suppose that we have defined the category $N\text{-}k\text{-}Grd\,\mathbb{B}$ of the normalized k-groupoids as far as level $n-1$ as well as a forgetful functor $\bar{F}_k : N\text{-}k\text{-}Grd\,\mathbb{B} \to k\text{-}Grd\,\mathbb{B}$.

Definition 3.1. A normalized n-groupoid in $\mathbb{B}$ is a n-groupoid whose underlying $(n-1)$-groupoid is normalized, and which is endowed with a given splitting σ_n (the normalization map at level n) of the terminal map $\tau_n : X_n \to g_n(X_{n-1})$ in the fibre of X_n.

We obtain in this way the category $N\text{-}n\text{-}Grd\,\mathbb{B}$ of the normalized n-groupoids whose morphisms are the n-functors which preserve the normalizations at each level. There is also a functor $\bar{F}_n : N\text{-}n\text{-}Grd\,\mathbb{B} \to n\text{-}Grd\,\mathbb{B}$ associating X_n to (X_n, σ_n).

Let us denote by $F_n : N\text{-}(n-1)\text{-}Grd\,\mathbb{B} \to n\text{-}Grd\,\mathbb{B}$ the functor $g_n.\bar{F}_{n-1}$. Then the category $N\text{-}n\text{-}Grd\,\mathbb{B}$ is defined by the following pullback:

$$\begin{tikzcd}
N\text{-}n\text{-}Grd\,\mathbb{B} \arrow[r, "\tilde{F}_n"] \arrow[d, "\tau_{n-1}"'] & Pt_{n-1}(n\text{-}Grd\,\mathbb{B}) \arrow[d, "p"] \\
N\text{-}(n-1)\text{-}Grd\,\mathbb{B} \arrow[r, "F_n"] & n\text{-}Grd\,\mathbb{B}
\end{tikzcd}$$

where $Pt_{n-1}(n\text{-}Grd\,\mathbb{B})$ denotes the category whose objects are the split hypomorphisms with respect to the fibration $(\ )_{n-1} : n\text{-}Grd\,\mathbb{B} \to (n-1)\text{-}Grd\,\mathbb{B}$.

It is shown in [5] that F_n has a right adjoint U_{n-1} that is monadic, which means that $n\text{-}Grd\,\mathbb{B}$ is the category of algebras of the monad $(T_{n-1}, \lambda_{n-1}, \mu_{n-1})$ on $N\text{-}(n-1)\text{-}Grd\,\mathbb{B}$ generated by the adjunction (U_{n-1}, F_n). Consequently, according to Theorem 2.2 and Proposition 2.3, we have as announced:

Theorem 3.1. *The category $\check{T}_n(Pt\,\mathbb{B})$ is isomorphic to $N\text{-}n\text{-}Grd\,\mathbb{B}$.*

Moreover, the right adjoint $\tilde{U}_n$ of $\tilde{F}_n$ is again monadic [5, Theorems 25 and 31] and the pairs $(\tilde{U}_n, \tilde{F}_n)$ and $(U_0, F_0) : U_0 : Alg_{n-1}\,T_0 \to Pt_{n-1}(n\text{-}Grd\,\mathbb{B})$ are distributively cohesive [5, Definition 7]. Consequently, the canonical comparison functor $\varphi_{n+1} : Grd_{n-1}(n\text{-}Grd\,\mathbb{B}) = (n+1)\text{-}Grd\,\mathbb{B} \to Alg\,\check{T}_n$ described in Remark 2.2 is an equivalence of categories.

4. The functor nerve for n-groupoids

Let us start from a pointed fibration $\tau : \mathbb{E} \to \mathbb{B}$ with a left exact monad (T, λ, μ) on $\mathbb{E}$ which is connected and transverse to τ. Then we have the projection $W_1 : \check{T}_1\mathbb{E} = \hat{T}\mathbb{E} \to T\mathbb{E} = T_1\mathbb{E}$ which is a strict morphism of monads, preserves the hypomorphisms and furthermore is left exact. The two first conditions determine a functor $TW_1 : T(\check{T}_1 E) \to T(T_1\mathbb{E}) = T^2\mathbb{E}$ and the third one makes it a strict morphism of monads which is itself left exact. Set $W_2 = TW_1.W$ making the following diagram commutative:

$$\begin{tikzcd}
\check{T}_2\mathbb{E} = \hat{T}(\check{T}_1\mathbb{E}) \arrow[rr, "w"] \arrow[dr, "\check{\tau}_1"'] & & T(\check{T}_1\mathbb{E}) \arrow[rr, "Tw_1"] \arrow[dl, "\tau_1"'] & & T_2\mathbb{E} \arrow[dl, "\tau_1"'] \\
& \check{T}_1\mathbb{E} \arrow[rr, "w_1"'] \arrow[ul, shift right, "\check{k}_2"'] \arrow[ur, shift right, "k_2"'] & & T_1\mathbb{E} \arrow[ur, shift right, "k_2"'] &
\end{tikzcd}$$

Let us suppose we have defined a left exact functor $W_{n-1} : \check{T}_{n-1}\mathbb{E} \to T_{n-1}\mathbb{E}$ which is a strict morphism of monads and such that the following diagram commutes:

$$\begin{tikzcd}
\check{T}_{n-1}\mathbb{E} \arrow[r, "W_{n-1}"] \arrow[d, shift right, "\check{\tau}_{n-2}"'] & T_{n-1}\mathbb{E} \arrow[d, shift right, "\tau_{n-2}"'] \\
\check{T}_{n-2}\mathbb{E} \arrow[r, "W_{n-2}"'] \arrow[u, shift right, "\check{k}_{n-1}"'] & T_{n-2}\mathbb{E}. \arrow[u, shift right, "k_{n-2}"']
\end{tikzcd}$$

Then the functor $W_n = TW_{n-1}.W$ satisfies the same conditions:

$$\check{T}_n\mathbb{E} = T(\check{T}_{n-1}\mathbb{E}) \xrightarrow{W} T(\check{T}_{n-1}\mathbb{E}) \xrightarrow{TW_{n-1}} T(T_{n-1}\mathbb{E}) = T_n\mathbb{E}.$$

When the starting fibration τ is $p : Pt\,\mathbb{B} \to B$, we thus get a left exact strict morphism of monads:

$$W_n : N\text{-}n\text{-}Grd\,\mathbb{E} \to S\text{-}Simpl_n\,\mathbb{E}$$

whose extension to the category of algebras produces a left exact functor:

$$v_{n+1} : Alg\,W_n.\varphi_{n+1} : (n+1)\text{-}Grd\,\mathbb{B} \approxeq Alg\,\check{T}_n \to K_{n+1}\mathbb{E}$$

which is the nerve functor for $(n+1)$-groupoids. It is described precisely in [6].

Proposition 4.1. *The functor W_n reflects hypomorphisms and identity maps.*

Proof. By induction. It is the case for $W_1 = W : \hat{T}\mathbb{E} \to T\mathbb{E}$ (Proposition 1.3).

Let us suppose it is true for all k, $1 \leq k \leq n-1$. It is straightforward that if $F : \mathbb{E} \to \mathbb{E}'$ reflects identity maps then $TF : T\mathbb{E} \to T\mathbb{E}'$ reflects hypomorphisms and identity maps. Thus by the inductive hypotheses, TW_{n-1} reflects these two kinds of morphisms and therefore $W_n = TW_{n-1}.W$ has the same property. □

5. Characterization of the nerve for *n*-groupoids

We are now going to study how to recognize among the simplicial objects those which are actually (the nerve of) a n-groupoid. We shall begin by recalling the category $K_n\mathbb{B}$ of algebras for the monad $(T_{n-1}, \lambda_{n-1}, \mu_{n-1})$ on $S\text{-}Simpl_{n-1}\,\mathbb{B}$.

Given a n-family of maps $d_i : X \to Y$, $0 \leq i \leq n-1$, the n-simplicial kernel of this family is the data of an object $X[d_0, \dots, d_{n-1}]$ together with a $(n+1)$-family of maps $p_i : X[d_0, \dots, d_{n-1}] \to X$, $0 \leq i \leq n$, which is universal with respect to the simplicial identities:

$$d_i.p_j = d_{j-1}.p_i, \qquad 0 \leq i < j \leq n.$$

Given a n-simplicial object $\underline{X}_n$, its initial $(n+1)$-horn $\bigwedge \underline{X}_n$ is just the n-simplicial kernel $X_n[d_0, \dots, d_{n-1}]$ (the map d_n is excluded) [6]. It is the universal open $(n+1)$-simplex whose last face is missing.

Definition 5.1. A Kan operation on a n-simplicial object $\underline{X}_n$ is a map $t : \bigwedge \underline{X}_n \to X_n$ satisfying the following axioms:

(1) *Contiguity axioms*: To a $(n+1)$-horn, the Kan operation t associates a top face such that

$$d_i.t = d_n.p_i, \quad 0 \leq i \leq n,$$
$$t.s_i = s_i.d_n, \quad 0 \leq i \leq n-1.$$

(2) "*Unitary*" *axiom*:

$$t.s_n = 1_{X_n}.$$

(3) "*Associativity*" *axiom*:

$$t.t[d_n] = t.p_n[d_{n-1}](= t.p_{n+1}).$$

In other words, the operation satisfies the axioms of a hypergroupoid in the sense of Glenn [13], except that the operation is only defined at the last level and uniquely for the initial horns (and not for any kind of horn). Thus there is a considerable saving of data.

A morphism of Kan operations is a n-simplicial map which preserves the $(n+1)$-ary operations. Thus we have a category denoted by $K_n\mathbb{B}$.

Remark 5.1. (1) A 1-simplicial object is just a reflexive graph. A Kan operation on this reflexive graph is just a groupoid structure, see [3], with $t(f,g) = g.f^{-1}$. Therefore $K_1\mathbb{B} = Grd\,\mathbb{B}$.

(2) It is shown in [6], that when $\mathbb{B}$ is additive there is always a unique Kan operation on any n-simplicial object.

There is an obvious forgetful functor $W_n : K_n\mathbb{B} \to Simpl_n\,\mathbb{B}$ defined by $W_n(\underline{X}_n, t) = \underline{X}_n$. If we denote by $\underline{U}_n : Simpl_n\,\mathbb{B} \to S\text{-}Simpl_{n-1}\,\mathbb{B}$ the functor obtained by shifting the index by one degree and cancelling the last face operation, we have a functor:

$$\bar{U}_{n-1} = \underline{U}_n.W_n : K_n\mathbb{B} \to S\text{-}Simpl_{n-1}\,\mathbb{B}.$$

Then it is shown in [6] that:

Theorem 5.1. *The category $K_n\mathbb{B}$ is the category of algebras for the monad $(T_{n-1}, \lambda_{n-1}, \mu_{n-1})$ on S-$Simpl_{n-1}\,\mathbb{B}$ and $\bar{U}_{n-1}$ is the canonical forgetful functor from the category of algebras to the base category.*

As we recalled, the category $K_n\mathbb{B}$ is a full subcategory of the category $Simpl\,\mathbb{B}$ since $(T_{n-1}, \lambda_{n-1}, \mu_{n-1})$ is the restriction on $S\text{-}Simpl_{n-1}\,\mathbb{B}$ of the monad (T, λ, μ) on $S\text{-}Simpl\,\mathbb{B}$. A simplicial object $\underline{X}$ "is" a Kan operation on a n-simplicial object $\underline{X}_n$ if and only if:

$$X_{n+1} = X_n[d_0,\dots,d_{n-1}],\ X_{n+k+1} = X_{n+k}[d_0,\dots,d_{n+k}] \qquad \forall k,\ 1 \le k,$$

see [6, Proposition 12].

We just saw that the nerve of a n-groupoid is endowed with a Kan operation which reflects the composition of the n-cells, but clearly it has something more which reflects the composition of the cells of lower levels.

Definition 5.2. A θ_n-complex in a category $\mathbb{B}$ is a Kan operation $(\underline{X}_n, t)$ on a n-simplicial object together with a family of maps $\theta_{k+1} : X_k[d_0,\dots,d_{k-1}] \to X_{k+1}, 1 \le k \le n-1$, such that

(1) θ_{k+1} is a section of the canonical factorization:

$$[d_0,\dots,d_k] : X_{k+1} \to X_k[d_0,\dots,d_{k-1}],$$

(2) $\theta_{k+1}.s_i = s_i,\ 0 \le i \le k$,

(3) $d_{k+2}.\theta_{k+2}.\theta_{k+1}[1] = \theta_{k+1}.(d_{k+1}.\theta_{k+1})[d_k],\ 1 \le k \le n-2$,

(4) $t.\theta_n[1] = \theta_n.(d_n.\theta_n)[d_{n-1}]$.

In other words, θ_{k+1} is a process which associates to each open $(k+1)$-simplex whose last face is missing an actual $(k+1)$-simplex with the same faces (axiom (1)).

By axiom (2), the degeneracies are simplexes of this kind, and by axioms (3) and (4), this process is coherent with increasing of index.

More precisely when $\mathbb{B} = Sets$, the map θ_{k+1} being a monomorphism, it is possible to identify $X_k[d_0,\dots,d_{k-1}]$ with its image T_{k+1} in X_{k+1}. We shall call T_{k+1} the set of thin elements of degree $(k+1)$. Then axiom (1) means that: every initial horn has a unique thin filler, axiom (2) that every degenerate element is thin, axiom (3) that if all the faces but the top one of a thin element are thin, then the top face is again thin.

Axiom (4) means that this last property holds if we consider the n-initial horns completed by the top face given by the Kan operation as a thin element of degree $(n+1)$.

The terminology here is borrowed from Dakin [11], where the word thin qualifies those simplexes which represent the cells which are actually given by an identity cell. But contrary to Dakin, the previous conditions (1), (3) and (4) are only demanded for initial horns and not any kind of horns.We have here a considerable saving of data which allows to define thin elements in full generality and in any internal context.

Let us exhibit axiom (3) by the following diagram:

$$\begin{tikzcd}
X_k^2[d_0,\dots,d_{k-1}] \arrow[rr, "\Theta_{k+1}[1]"] \arrow[dr, "(d_{k+1}.\Theta_{k+1})\,[d_k]"'] & & X_{k+1}[d_0,\dots,d_k] \arrow[rr, "\Theta_{k+2}"] & & X_{k+2} \arrow[dl, "d_{k+2}"] \\
& X_k[d_0,\dots,d_{k-1}] \arrow[rr, "\Theta_{k+1}"'] & & X_{k+1} \arrow[dl, "d_{k+1}"] & \\
& & X_k & &
\end{tikzcd}$$

which is based upon the property (see [6, Proposition 2, Corollary 1]) following which the iterated k-simplicial kernel $X_k^2[d_0,\dots,d_{k-1}]$ of the k-family $(d_0,\dots,d_{k-1})$ is the $(k+1)$-simplicial kernel of the $(k+1)$-family $(p_0,\dots,p_k)$, $p_i : X_k[d_0,\dots,d_{k-1}] \to X_k$ and that consequently the map θ_{k+1} satisfying axiom (1), determines the factorization $\theta_{k+1}[1]$ at the level of the $(k+1)$-simplicial kernels. The map $(d_{k+1}.\theta_{k+1})[d_k]$ denotes the factorization induced by $d_{k+1}.\theta_{k+1}$ at the level of the k-simplicial kernels. Axiom (4) means that θ_{n+1} is an identity map.

Remark 5.2. In the additive situation, i.e. when $\mathbb{E}$ is additive, any simplicial kernel of a split simplicial object is a simplicial cokernel (see [6]). Thus any n-simplicial object has a unique canonical structure of θ_n-complex. As a consequence, in this case, the category of internal n-groupoids in $\mathbb{E}$ is equivalent to the category of n-simplicial objects in $\mathbb{E}$ through the nerve functor, see also [3].

We shall denote by θ_n-$\mathbb{B}$ the category whose objects are the θ_n-complexes in $\mathbb{B}$ and the maps the morphisms of Kan operations which preserve the family of maps θ_k, in other words which preserve the thin elements.

By definition, there is a forgetful functor $h_n : \theta_n\text{-}\mathbb{B} \to K_n\mathbb{B}$. Moreover, given $(\underline{X}_n, t, \theta_{k+1}, 1 \leq k \leq n-1)$ a θ_n-complex, then $(\tau_{n-1}\underline{X}_n, d_n.\theta_n, \theta_k, 1 \leq k \leq n-2)$

is a θ_{n-1}-complex, where τ_{n-1} denotes the functor $Simpl_n\,\mathbb{B} \to Simpl_{n-1}\,\mathbb{B}$ which erases the higher level. The map $d_n.\theta_n$ is indeed a Kan operation on τ_{n-1} since the associativity is given by

$$d_n.\theta_n.(d_n.\theta_n)[d_{n-1}] = d_n.t.\theta_n[1] = d_n.d_n.\theta_n[1] = d_n.\theta_n.p_n.$$

Let us denote by $r_{n-1} : \theta_n\text{-}\mathbb{B} \to \theta_{n-1}\text{-}\mathbb{B}$ this functor and by $k_n : K_{n-1}\mathbb{B} \to K_n\mathbb{B}$ the extension to the categories of algebras of the functor $k_{n-1} : S\text{-}Simpl_{n-2}\,\mathbb{B} \to S\text{-}Simpl_{n-1}\,\mathbb{B}$ (Proposition 1.1, Remark 1.3). Then thanks to axiom (4), the map θ_n determines a natural transformation $\bar{\theta}_n : k_n.h_{n-1}.r_{n-1} \Rightarrow h_n$. Moreover, the map underlying $\bar{\theta}_n$ (considered as a morphism of algebras of the monad $(T_{n-1}, \lambda_{n-1}, \mu_{n-1})$) in $S\text{-}Simpl_{n-1}\,\mathbb{B}$ is a hypomorphism with respect to $\tau_{n-2} : S\text{-}Simpl_{n-1}\,\mathbb{B} \to S\text{-}Simpl_{n-2}\,\mathbb{B}$.

$$\begin{tikzcd}
\theta_n\mathbb{B} \arrow[r, "h_n"] \arrow[d, "r_{n-1}"'] \arrow[dr, phantom, "\bar{\theta}_n \Uparrow"] & K_n\mathbb{B} \\
\theta_{n-1}\mathbb{B} \arrow[r, "h_{n-1}"'] & K_{n-1}\mathbb{B} \arrow[u, "k_n"']
\end{tikzcd}$$

The following proposition will give us an inductive description of the categories $\theta_n\text{-}\mathbb{B}$:

Proposition 5.2. *The category $\theta_n\text{-}\mathbb{B}$ is the hypo-comma category of the functors $k_n.h_{n-1}$ and $Id\,K_n\mathbb{B}$.*

Proof. This means that $\theta_n\text{-}\mathbb{B}$ is the comma category with its natural transformation a hypomorphism. The proof is straightforward. Let $(\underline{Z}_{n-1}, \tau, \theta_k, 1 \le k \le n-2)$ be an object in $\theta_{n-1}\text{-}\mathbb{B}$ and (X_n, t) an object in $K_n\mathbb{B}$ with a hypomorphism $\gamma : k_n.h_{n-1}(\underline{Z}_{n-1}, \tau, \theta_k) \to (X_n, t)$. That means γ_k is an identity map and thus $Z_k = X_k\ \forall k,\ 0 \le k \le n-1$, and that γ_n is a map: $X_{n-1}[d_0, \dots, d_{n-2}] \to X_n$ satisfying

(1) $d_n.\gamma_n = \tau$,

(2) $t.\gamma_n[1] = \gamma_n.\tau[d_{n-1}]$.

Let us set $\theta_n = \gamma_n$, then $(X_n, t, \theta_k, 0 \le k \le n-1)$ is an object of $\theta_n\text{-}\mathbb{B}$ since:

$$t.\theta_n[1] = t.\gamma_n[1] = \gamma_n.\ \tau[d_{n-1}] = \theta_n.(d_n\theta_n)[d_{n-1}].$$

We claim that the categories $n\text{-}Grd\,\mathbb{B}$ and $\theta_n\text{-}\mathbb{B}$ are the same. In other words, a θ_n-complex gives a simplicial presentation of an n-groupoid. The end of this paper will be devoted to the proof of this claim.

Indeed, the commutation $W_n.\check{k}_n = k_{n-1}.W_{n-1}$ of Section 4 extends to the category of algebras since all the functors are strict morphisms of monads. Now, since the fibration $\check{\tau}_{n-1} : \check{T}_n\mathbb{B} \to \check{T}_{n-1}\mathbb{B}$ is pointed, $\check{\tau}_{n-1}$ is a lax morphism of monads and extends to a functor $Alg\,\check{\tau}_n$ which is a right adjoint to $Alg\,\check{k}_n$. If we denote by $\sigma : Alg\,\check{k}_n.Alg\,\check{\tau}_{n-1} \Rightarrow 1$ the associated natural transformation which is clearly a hypomorphism, then we get a hypomorphic natural transformation $Alg\,W_n.\sigma : Alg\,k_n.Alg\,W_{n-1}.Alg\,\check{\tau}_{n-1} = Alg\,W_n.Alg\,\check{k}_n.Alg\,\check{\tau}_{n-1} \Rightarrow Alg\,W_n$:

$$\begin{tikzcd}
Alg\,\check{T}_n \arrow[r, "Alg\,W_n"] \arrow[d, "Alg\,\check{\tau}_{n-1}"'] \arrow[dr, phantom, "\Uparrow Alg\,W_n\sigma"] & Alg\,T_n \\
Alg\,\check{T}_{n-1} \arrow[r, "Alg\,W_{n-1}"'] & Alg\,T_{n-1} \arrow[u, "Alg\,k_n"']
\end{tikzcd}$$

Our claim will rely on the following proposition:

Proposition 5.3. *When the extension functor $Alg\,W_n$ is discretely cofibrant on the hypomorphisms, the previous diagram is the hypocomma object of $Alg\,k_n.Alg\,W_{n-1}$ and $Id\,Alg\,T_n$.*

Proof. The condition means that given an object U in $Alg\,\check{T}_n$ and a hypomorphism $\gamma : Alg\,W_n(U) \to V$, there is a unique $\bar{\gamma} : U \to \bar{V}$ such that $Alg\,W_n(\bar{\gamma}) = \gamma$. But W_n reflects the hypomorphisms (Proposition 4.1). Thus $\bar{\gamma}$ is a hypomorphism and $Alg\,\check{\tau}_{n-1}(\bar{V}) = Alg\,\check{\tau}_{n-1}(U)$. Now an object of the hypocomma category is just a pair (Z, γ) of an object Z in $Alg\,\check{T}_{n-1}$ and a hypomorphism $\gamma : Alg\,k_n.Alg\,W_{n-1}(Z) \to X$ in $Alg\,T_n$. But the domain of γ is also $Alg\,W_n.Alg\,\check{k}_n(Z)$, and there is a unique $\bar{\gamma} : Alg\,\check{k}_n(Z) \to \bar{X}$ in $Alg\,\check{T}_n$ such that $Alg\,W_n(\bar{\gamma}) = \gamma$. Thus $Alg\,\check{\tau}_{n-1}(\bar{X}) = Z$ and necessarily $\bar{\gamma} = \sigma\bar{X}$. Therefore $\gamma = Alg\,W_n(\bar{\gamma}) = Alg\,W_n(\sigma\bar{X})$. □

Theorem 5.4. *In the situation of Proposition* 5.3, *then θ_n-$\mathbb{B}$ and $Alg\,\check{T}_n$ are isomorphic.*

Proof. By induction. It is clear that θ_1-$\mathbb{B} = K_1\mathbb{B}$. Let us suppose there is an isomorphism $\varphi_k : \theta_k\text{-}\mathbb{B} \to Alg\,\check{T}_k$ for all $k \leq n-1$ such that $Alg\,W_k.\varphi_k = h_k$. Now $Alg\,T_k = K_k\mathbb{B}$. So let us consider the following diagram:

$$\begin{tikzcd}
\Theta_n\text{-}\mathbb{B} \arrow[rr, "h_n"] \arrow[dr, dashed, "\varphi_n"] \arrow[dd, "r_{n-1}"'] & & K_n\mathbb{B} \\
& Alg\check{T}_n \arrow[ur, "Algw_n"'] \arrow[d, "Alg\check{\tau}_{n-1}"'] & \\
\Theta_{n-1}\text{-}\mathbb{B} \arrow[r, "\varphi_{n-1}"] \arrow[rr, bend right, "h_{n-1}"'] & Alg\check{T}_{n-1} \arrow[r, "Algw_{n-1}"] & K_{n-1}\mathbb{B} \arrow[uu, "k_n"']
\end{tikzcd}$$

The two squares determining hypocommacategories, there is a unique factorization φ_n such that the following square is a pullback:

$$\begin{tikzcd}
\theta_n\mathbb{B} \arrow[r, "\varphi_n"] \arrow[d, "r_{n-1}"'] & Alg\check{T}_n \arrow[d, "Alg\check{\tau}_{n-1}"] \\
\theta_{n-1}\mathbb{B} \arrow[r, "\varphi_{n-1}"'] & Alg\check{T}_{n-1}
\end{tikzcd}$$

Since φ_{n-1} is an isomorphism, φ_n is an isomorphism. □

Consequently: The category n-$Grd\,\mathbb{B}$ and θ_n-$\mathbb{B}$ are equivalent by $n\text{-}Grd\,\mathbb{B} \xrightarrow{\sim} Alg\check{T}_n \xrightarrow[\varphi_n^{-1}]{} \theta_n\text{-}\mathbb{B}$. Now, we must show that $Alg\,W_n$ fullfils the condition of Proposition 5.3.

6. The nerve functor is discretely cofibrant on the hypomorphisms

The monad (T_1, λ_1, μ_1) on $T\mathbb{E}$ fullfils a property we have not yet used.

Definition 6.1. Given a left exact fibration $\tau : \mathbb{E} \to \mathbb{B}$ and (T, λ, μ) a monad on $\mathbb{E}$, the monad is called equable with respect to τ (or τ-equable), when it is connected to τ and for any object X in $\mathbb{E}$ the following diagram is a pullback in $\mathbb{B}$:

$$\begin{tikzcd}
\tau TX \arrow[d, "\tau T\omega X"'] & & \tau T^2X \arrow[ll, "\tau\mu X"'] \arrow[d, "\tau T\omega TX"] \\
T_B\tau X \arrow[r, "\mu_B\tau X"'] & T_B^2\tau X \arrow[r, "T_B\tau T\omega X"'] & T_B\tau TX
\end{tikzcd}$$

Example 6.1. The monad (T_1, λ_1, μ_1) on $T\mathbb{E}$ is equable with respect to $\tau_0 : T\mathbb{E} \to \mathbb{E}$, in the same way as $(\hat{T}_1, \hat{\lambda}_1\hat{\mu}_1)$ on $\hat{T}\mathbb{E}$ with respect to $\hat{\tau}_0 : \hat{T}\mathbb{E} \to \mathbb{E}$.

Our main technical result will be the following:

Proposition 6.1. *Given a left exact transverse and equable monad (T, λ, μ) on $\mathbb{E}$ with respect to $\tau : \mathbb{E} \to \mathbb{B}$, and any split hypomorphism (d,s), $d : X \to Y$ in $\mathbb{E}$, the factorization through the pullback determined by the following square is split by a map ψ such that $\psi.sd = T^2s$ and $\mu TX.T\psi.\bar{\psi} = \psi.n_0Nd$.*

$$\begin{tikzcd}
TX \arrow[d, "Td"'] & Nd \arrow[l, "n_0 d"] \arrow[d, "nd"'] \arrow[r, bend right, "\Psi"'] & T^2X \arrow[ll, bend right, "\mu X"'] \arrow[l, dashed, "\varphi"'] \arrow[dl, "T^2d"] \\
TY & T^2Y \arrow[l, "\mu Y"] &
\end{tikzcd}$$

where sd is the splitting of nd induced by Ts, and $\bar{\psi} : N^2d \to TNd$ is the factorization induced by ψ.

Proof. The map μX being cartesian, it is sufficient to exhibit a factorization at the level of the images by the fibration τ. So let us consider the following diagram:

$$\begin{tikzcd}
\tau TX \arrow[d, "\tau Td"] \arrow[dd, bend right, "\tau T\omega X"'] & & \tau Nd \arrow[ll, "\tau n_0 d"'] \arrow[d, "\tau nd"] \arrow[dll, phantom, "*"] & \tau T^2X \arrow[lll, bend right, "\tau\mu X"'] \arrow[l, "\tau\varphi"] \arrow[dd, "\tau T\omega TX"] \\
\tau TY \arrow[d, "\tau T\omega Y"] & & \tau T^2Y \arrow[ll, "\tau\mu Y"'] \arrow[d, "\tau T\omega T_Y"] \arrow[dll, phantom, "**"] & \\
T_B\tau Y & T_B^2\tau Y \arrow[l, "\mu_B\tau Y"'] & T_B\tau TY \arrow[l, "T_B\tau T\omega Y"'] & T_B\tau TX \arrow[l, "T_B\tau T_d"'] \arrow[ll, bend left, "T_B\tau T\omega X"]
\end{tikzcd}$$

The lower square ** and the outer square are pullbacks since the monad is equable, the square * is a pullback by definition of Nd, consequently the right-hand square is a pullback. Therefore, the section $T_B\tau Ts$ of $T_B\tau Td$ produces a section of $\tau\varphi$, denoted by $\tau\psi$, since the equality $\tau\mu X.\tau\psi = \tau n_0 d.\tau\varphi.\tau\psi = \tau n_0 d$, produces a unique $\psi : Nd \to T^2X$ above $\tau\psi$ such that $\mu X.\psi = nd$. As a consequence, the map ψ is cartesian. Furthermore, we have $\varphi.\psi = 1$ since nd is cartesian as a pullback of a cartesian map, whence $T^2d.\psi = nd$. That $\psi.sd = T^2s$ is checked by composition with the cartesian map μX and by projection by τ through the universal property of τT^2X. The equality $\mu TX.T\psi.\bar{\psi} = \psi.n_0Nd$ is obtained in the same way, the difficult part of this being given by

$$\begin{aligned}
\tau T\omega TX.\tau\mu TX.\tau T\psi.\tau\bar{\psi} &= \mu_B\tau TX.T_B\tau T\omega TX.\tau T\omega T^2X.\tau T\psi.\tau\bar{\psi} \\
&= \mu_B\tau TX.T_B\tau T\omega TX.T_B\tau\psi.\tau T\omega Nd.\tau\bar{\psi} \\
&= \mu_B\tau TX.T_B^2\tau Ts.T_B\tau T\omega TY.T_B\tau nd.\tau T\omega Nd.\tau\bar{\psi} \\
&= T_B\tau Ts.\mu_B\tau TY.T_B\tau T\omega TY.\tau T\omega T^2Y.\tau Tnd.\tau\bar{\psi}
\end{aligned}$$

$$= T_B\tau Ts.\tau T\omega TY.\tau\mu TY.\tau n^2 d$$

$$= T_B\tau Ts.\tau T\omega TY.\tau nd.\tau n_0 Nd = \tau T\omega TX.\tau\psi.\tau n_0 Nd.$$

We are now going to establish the effect of the equability of the monad (T, λ, μ) on the monads (T_1, λ_1, μ_1) and $(\hat{T}_1, \hat{\lambda}_1 \hat{\mu}_1)$. $\square$

Definition 6.2. Given a diagram of the following type, in which the square commutes and $f.f_0 = f.f_1$:

$$\begin{tikzcd}
D \arrow[d, "h"'] & C \arrow[l, "g"'] \arrow[d, "k"] & \\
B & A \arrow[l, "f"] & T \arrow[l, shift right, "f_0"'] \arrow[l, shift left, "f_1"]
\end{tikzcd}$$

we shall call the kernel pair of g relative to (f_0, f_1), the data of a pair

$$p_0, p_1 : P \rightrightarrows C$$

and a morphism $\pi : P \to T$, universal with respect to the identities: $g.p_0 = g.p_1$ and $k.p_i = f_i.\pi$, $0 \le i \le 1$.

When the map h is an isomorphism, the relative kernel pair (p_0, p_1) is just the joint pullback of (f_0, f_1) along k. When the left-hand square is itself a pullback then the relative kernel pair is obtained by the pullback of k along any of the f_i.

Theorem 6.2. *Given an object $\underline{X}_1$ in $T\mathbb{E}$, a splitting s_1 of $\omega\underline{X}_1 : \underline{X}_1 \to k_1\tau_0\underline{X}_1$ determines a splitting γs_1 of the canonical factorization $\rho\underline{X}_1 : T_1^3\underline{X}_1 \to P(\underline{X}_1)$ through the kernel pair of $\mu\underline{X}_1$ relative to $(\mu_1 T_1 k_1\tau_0\underline{X}_1, T_1\mu_1 k_1\tau_0\underline{X}_1)$ such that*
(1) $\gamma s_1.\overline{T_1^2 s_1} = T_1^3 s_1$, where $\overline{T_1^2 s_1}$ is the factorization induced by $T_1^2 s_1$,
(2) $\gamma s_1.s_{-1} = \lambda_1 T_1^2\underline{X}_1$, $\gamma s_1.s_0 = T_1\lambda_1 T_1\underline{X}_1$, $\gamma s_1.s_1 = T_1^2\lambda_1\underline{X}_1$,
(3) $\mu_1 T_1^2\underline{X}_1.\gamma^2 s_1 = \gamma s_1.p_0$, $T_1\mu_1 T_1\underline{X}_1.\gamma^2 s_1 = \gamma s_1.p_1$, $T_1^2\mu_1 T_1\underline{X}_1.\gamma^2 s_1 = \gamma s_1.p_2$.

Proof. Let us consider the following diagram:

$$\begin{tikzcd}
 & & & T_1^3\underline{X}_1 \arrow[dll, shift right, "T_1\mu_1\underline{X}_1"'] \arrow[dll, shift left, "\mu_1 T_1 X_1"] \arrow[dl, "\rho\underline{X}_1"'] \arrow[dd, "T_1^3\omega X_1"] \\
T_1\underline{X}_1 \arrow[d, "T_1\omega\underline{X}_1"'] & T_1^2\underline{X}_1 \arrow[l, "\mu_1\underline{X}_1"'] \arrow[d, "T_1^2\omega\underline{X}_1"] & P(X_1) \arrow[l, shift right, "p_1"'] \arrow[l, shift left, "p_0"] \arrow[dr, "\pi"] & \\
T_1 k_1\tau_0\underline{X}_1 & T_1^2 k_1\tau_0\underline{X}_1 \arrow[l, "\mu_1 k_1\tau_0\underline{X}_1"] & & T_1^3 k_1\tau_0\underline{X}_1 \arrow[ll, shift right, "T_1\mu_1 k_1\tau_0\underline{X}_1"'] \arrow[ll, shift left, "\mu_1 T_1 k_1\tau_0\underline{X}_1"]
\end{tikzcd}$$

The map $\mu_1\underline{Z}$ being cartesian for any $\underline{Z}$, according to the previous remarks concerning the relative kernel pairs, the maps p_0 and p_1 are cartesian. Consequently, $\rho\underline{X}_1$ is cartesian and will be split, if and only if its image by τ_0 is split.

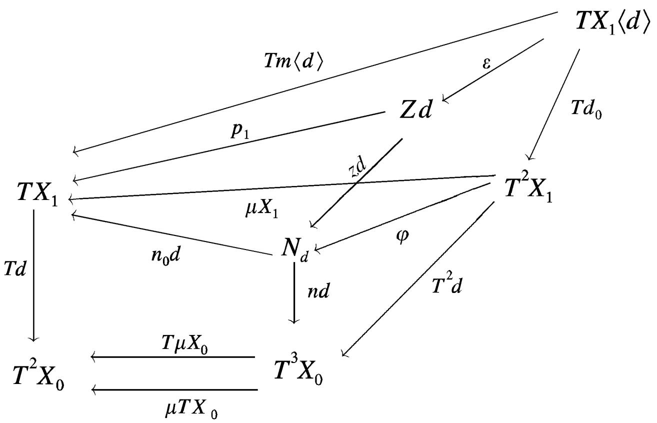
Now $\tau_0 T_1^2\omega\underline{X}_1 = T\tau_0 T_1\omega\underline{X}_1.\tau_0 T_1\omega T_1\underline{X}_1$ and the following square is a pullback:

$$\begin{tikzcd}
\tau_0 T_1\underline{X_1} \arrow[d, "\tau_0 T_1\omega\underline{X_1}"'] & & \tau_0 T_1^2\underline{X_1} \arrow[ll, "\tau_0\mu_1\underline{X_1}"'] \arrow[d, "\tau_0 T_1\omega T_1\underline{X_1}"] \\
T\tau_0\underline{X_1} & T^2\tau_0\underline{X_1} \arrow[l, "\mu\tau_0\underline{X_1}"] & T\tau_0 T_1\underline{X_1} \arrow[l, "T\tau_0 T_1\omega\underline{X_1}"]
\end{tikzcd}$$

Consequently, it is sufficient to prove that the factorization $\varepsilon\underline{X}_1$ of the pair $(\mu\tau_0 T_1\underline{X}_0.$ $T\tau_0 T_1\omega T_1\underline{X}_1, T\tau_0\mu_1\underline{X}_1)$ through the joint pullback $Z\underline{X}_1$ of the pair $(\mu T\tau_0\underline{X}_1, T\mu\tau_0\underline{X}_1)$ along $T\tau_0 T_1\omega\underline{X}_1$ is split, since the following square is a pullback:

$$\begin{tikzcd}
\tau_0 P\underline{X_1} \arrow[d, "\overline{\tau_0 T_1\omega T_1 X_1}"'] & \tau_0 T_1^3 X_1 \arrow[l, "\tau_0\rho\underline{X_1}"'] \arrow[d, "\tau_0 T_1\omega T_1^2\underline{X_1}"] \\
Z\underline{X_1} & T\tau_0 T_1^2\underline{X_1} \arrow[l, "\varepsilon\underline{X_1}"]
\end{tikzcd}$$

Now given $\underline{X}_1 = (X_0, d, l)$ an object in $T\mathbb{E}$, the splitting s_1 of ωX_1 is given by a splitting s of d such that $l = s.\lambda X_0$. Thus $T\tau_0 T_1\omega\underline{X}_1 = Td$, $\mu T\tau_0\underline{X}_1 = \mu TX_0$, $T\mu\tau_0\underline{X}_1 = T\mu X_0$. So let us consider the joint pullback Zd of $(\mu TX_0, T\mu X_0)$ along Td:



The object Zd is obtained by the pullback zd of Td along $T\mu X_0.nd$. Consequently, the following square is a pullback:

$$\begin{tikzcd} Zd \arrow[d, "zd"'] & TX_1\langle d\rangle = T\tau_0T_1^2\underline{X}_1 \arrow[l, "\varepsilon"'] \arrow[d, "Td_0"] \\ Nd & T^2X_1 = T^2\tau_0T_1\underline{X}_1 \arrow[l, "\varphi"] \end{tikzcd}$$

By definition of an object of $T\mathbb{E}$, the map d is a hypomorphism. Its section s determines a section ψ_s, according to Proposition 6.1, which itself determines a section η_s of $\varepsilon\underline{X}_1$ such that $Td_0.\eta_s = \psi_s zd$. Again this section η_s determines the desired section $\tau_0\gamma s_1$ of $\tau_0\rho\underline{X}_1$ which satisfies $\tau_0T_1\omega T_1^2X_1.\tau_0\psi_s = \eta_s.\overline{\tau_0T_1\omega T_1X_1}$. Consequently, we have $T_1^2\omega T_1X_1.\gamma s_1 = \psi_s.zd.\overline{T_1\omega T_1X_1}$. The equation $\gamma s_1.T_1^2s_1 = T_1^3s_1$ comes from $\psi_s.s_d = T^2s$ by Proposition 6.1.

In the same way the equations $\gamma s_1.s_{-1} = \lambda_1T_1^2\underline{X}_1$ and $\gamma s_1.s_0 = T_1\lambda_1T_1\underline{X}_1$ come from $\psi_s.s_{-1} = \lambda T^2X_0$ and $\psi_s.s_0 = T\lambda TX_0$. That $\gamma s_1.s_1 = T_1^2\lambda_1\underline{X}_1$ is a consequence of the naturalities of the construction.

The map γs_1^2 will be determined by the following observations:

Let us consider the following pullback:

$$\begin{tikzcd} Zd \arrow[d, "zd"'] & Nz \arrow[l, "p_0"'] \arrow[d, "nz"] \\ Nd & N^2d \arrow[l, "n_0d"] \end{tikzcd}$$

Then the equality $T\mu X_0.nd.n_0d = T\mu X_0.\mu T^2X_0.n_0^2d = \mu TX_0.T^2\mu X_0.n_0^2d$ produces a unique map $p_2 : Nz \to Nd$ such that $n_0d.p_2 = p_1.p_0$, $nd.p_2 = T^2\mu X_0.n^2d.nz$. Then there is a unique factorization $\varphi_z : T^2\tau_0T_1^2\underline{X}_1 \to Nz$ such that $p_2.\varphi_z = \varphi.T^2\tau_0T_1\mu_1\underline{X}_1$ and $nz.\varphi_z = \varphi^2.T^2\tau_0T_1\omega T_1\underline{X}_1$.

Now the sections ψ_s and ψ_s^2 determine a unique section ψ_z of φ_z such that $T^2\tau_0\mu_1\underline{X}_1.\psi_z = \psi_s.p_2$ and $T^2\tau_0T_1\omega_1T_1\underline{X}_1.\psi_z = \psi_s^2.nz$. It is easy to check that $\mu\tau_0T_1^2\underline{X}_1.\psi_z = \eta_s.p_0$. Now if we consider the following pullback which defines the object Z^2d:

$$\begin{tikzcd} Zd \arrow[d, "zd"'] & & Z^2d \arrow[ll, "p_1"'] \arrow[d, "z^2d"] \\ Nd & N^2d \arrow[l, "n_1d"] & Nz \arrow[l, "nz"] \end{tikzcd}$$

there is a unique map $p_2 : Z^2d \to Zd$ such that $zd.p_2 = p_2.z^2d$ and $p_1.p_2 = p_1.p_1$.

There is a factorization $\varepsilon^2 : T\tau_0T_1^3\underline{X}_1 \to Z^2d$ such that the following square is a pullback:

$$\begin{tikzcd} Z^2d \arrow[d, "z^2d"'] & T\tau_0 T_1^3\underline{X_1} \arrow[l, "\varepsilon^2"'] \arrow[d, "T\tau_0 T\omega T_1^2\underline{X_1}"] \\ Nz & T^2\tau_0 T_1^2\underline{X_1} \arrow[l, "\varphi z"] \end{tikzcd}$$

and consequently the section ψ_z of φ_z determines a section η_s^2 of ε^2 satisfying

$$\mu\tau_0 T_1^2\underline{X}_1.T\tau_0 T_1\omega T_1^2\underline{X}_1.\eta_s^2 = \eta_s.p_0.z^2d,$$

$$T\tau_0\mu_1 T_1\underline{X}_1.\eta_s^2 = \eta_s.p_1,$$

$$T\tau_0 T_1\mu_1\underline{X}_1.\eta_s^2 = \eta_s.p_2.$$

Now the following square being a pullback:

$$\begin{tikzcd} P^2\underline{X_1} \arrow[d, "\overline{\overline{T_1\omega T_1 X_1}}"'] & T_1^4\underline{X_1} \arrow[l, "\rho^2 X_1"'] \arrow[d, "T_1\omega T^3\underline{X_1}"] \\ Z^2d & T\tau_0 T_1^3\underline{X_1} \arrow[l, "\varepsilon^2"] \end{tikzcd}$$

the section η_s^2 produces the section $\gamma^2 s_1$ satisfying axiom (3). $\square$

If we consider now an object $\underline{X}_1$ in $\hat{T}\mathbb{E}$, the section $\sigma\underline{X}_1$ of $\underline{\omega}X_1$ is natural and consequently the relation between $\hat{T}_1$ and T is natural. Thus:

Proposition 6.3. *For any object $\underline{X}_1$ in $\hat{T}\mathbb{E}$, there is a natural section $\gamma\underline{X}_1 : P(X_1) \to T_1^3 X_1$ of ρX_1 satisfying*
(1) $\gamma\underline{X}_1.\overline{\hat{T}_1^2\sigma\underline{X}_1} = \hat{T}_1\sigma\underline{X}_1$,
(2) $\gamma\underline{X}_1.s_{-1} = \hat{\lambda}_1\hat{T}_1^2\underline{X}_1$, $\gamma\underline{X}_1.s_0 = \hat{T}_1\hat{\lambda}_1\hat{T}_1\underline{X}_1$, $\gamma\underline{X}_1.s_1 = \hat{T}_1^2\hat{\lambda}_1\underline{X}_1$,
(3) $\hat{T}_1^2\hat{\mu}_1\underline{X}_1.\gamma\hat{T}_1\underline{X}_1.P\gamma\underline{X}_1 = \gamma\underline{X}_1.p_2$.

Proof. The two first properties follow from the previous proposition. The third one is the natural version of the third one of the previous proposition. Indeed, $\gamma\overline{T_1\underline{X}_1}.P\gamma\underline{X}_1$ is just $\gamma^2_{\sigma X_1}$ by the universal property of $\hat{T}_1^4\underline{X}_1$, and the fact that $\hat{T}_1\sigma\underline{X}_1 = \sigma T_1\underline{X}_1.k_1\tau_0\hat{T}_1\sigma\underline{X}_1$. Consequently the third property is just $T_1^2\mu_1 T_1\underline{X}_1.\gamma^2 s_1 = \gamma s_1.p_2$. $\square$

From this last result we are now going to derive a very important property of the monads $(\hat{T}_n, \hat{\lambda}_n, \hat{\mu}_n)$.

7. Normalized monads

Definition 7.1. A monad (T, λ, μ) on a category $\mathbb{E}$ is called normalized when the natural factorization $\alpha : T^3X \to T^2X[\mu X]$ is naturally split by a map βX such that
(1) $s_{-1}.\beta X = \lambda T^2 X$, $s_0.\beta X = T\lambda TX$, $s_1.\beta X = T^2\lambda X$ and,

(2) $T^2\mu X.\beta TX.\beta X[1] = \beta X.p_2$.

Of course, the fact that βX is a section of α is equivalent to: $\mu TX.\beta X = p_0$, $T\mu X.\beta X = p_1$.

Theorem 7.1. *Given a normalized monad, any algebra x on an object X determines a canonical structure of groupoid on the following graph*:

$$\begin{tikzcd} TX \arrow[r, "T\lambda TX"'] & T^2X \arrow[l, bend right, "Tx"'] \arrow[l, bend left, "\mu X"] \end{tikzcd}$$

which is normalized by the map λTX.

Proof. Let us consider the following diagram:

$$\begin{tikzcd} TX \arrow[r] & T^2X \arrow[l, bend right, "Tx"'] \arrow[l, bend left, "\mu X"] & & T^3X \arrow[ll, "T^2x"'] \\ & & & T^2X[\mu X] \arrow[ull, shift left, "p_1"'] \arrow[ull, shift right, "p_0"] \arrow[u, "\beta X"'] \end{tikzcd}$$

Then the map $T^2x.\beta X$ is a candidate to complete the groupoid structure since $\mu X.T^2x.\beta X = Tx.\mu T^2X.\beta X = Tx.p_0$ and $Tx.T^2x.\beta X = Tx.T\mu TX.\beta X = Tx.p_1$. We must check that the factorization δ_3 such that $p_i.\delta_3 = T^2x.\beta X.p_i$ $(0 \leq i \leq 1)$ satisfies also $T^2x.\beta X.\delta_3 = T^2x.\beta X.p_2$. But clearly $\delta_3 = T^2x[Tx].\beta X[1]$ and $T^2x.\beta X.T^2x[Tx].\beta X[1] = T^2x.T^3x.\beta TX.\beta X[1] = T^2x.T^2\mu X.\beta TX.\beta X[1] = T^2x.p_2$. $\square$

This kind of monad allows one to control the class of maps to which the algebras mappings belong. Let us call left proper a class Σ of maps containing the isomorphisms and being such that when g is in Σ, then $g.f$ is in Σ if and only if f is in Σ.

Theorem 7.2. *Let (T, λ, μ) be a normalized monad, tranverse to a left proper class Σ (i.e. λX and μX are in Σ), then any algebra structure $x : TX \to X$ is in Σ.*

Proof. The previous groupoid structure makes Tx equal up to isomorphism to μX and consequently implies that Tx is in Σ. So $Tx.\lambda TX = \lambda X.x$ is in Σ, as well as λX. Thus x is in Σ. $\square$

Example 7.1. The monad (T_0, λ_0, μ_0) on $Pt\,\mathbb{E}$ is not only normalized but normal (i.e αX is an isomorphism).

The class of cartesian maps with respect to p is left proper and the monad (T_0, λ_0, μ_0) is transverse to p. Thus any algebra structure of (T_0, λ_0, μ_0) is cartesian.

Theorem 7.3. *The monads (T_n, λ_n, μ_n) on N-$nGrd\,\mathbb{B}$ are normalized.*

Proof. By induction. We saw that (T_0, λ_0, μ_0) is normal. Let us suppose (T_k, λ_k, μ_k) normalized for all $k \leq n-1$. We know that N-n-$Grd\,\mathbb{B}$ is $\hat{T}_{n-1}(N\text{-}(n-1)\text{-}Grd\,\mathbb{B})$ and that the monad T_n is the monad $(\widehat{T_{n-1}})_1$. Furthermore the monad $(T_{n-1}, \lambda_{n-1}, \mu_{n-1})$ is equable. So let us consider the following diagram:

$$
\begin{tikzcd}
T_n\underline{X}_n \arrow[d, "T_n\omega\underline{X}_n"'] & T_n^2\underline{X}_n \arrow[l, "\mu_n\underline{X}_n"'] \arrow[d, "T_n^2\omega\underline{X}_n"'] & & & T_n^3\underline{X}_n \arrow[llll, shift right] \arrow[llll, shift left] \arrow[dl, shift right, "\rho(\underline{X}_n)"'] \arrow[ddd, "T_n^3\omega\underline{X}_n"] \\
& & & P(\underline{X}_n) \arrow[ur, shift right, "\gamma"'] \arrow[dl, shift right, "\bar{\alpha}"'] \arrow[ddr, "\pi"] & \\
& & T_n^2\underline{X}_n[\mu_n] \arrow[ull, shift left] \arrow[ull, shift right] \arrow[ur, shift right, "\bar{\beta}"'] \arrow[dd, "T_n^2\omega\underline{X}_n[T_n\omega X_n]"'] & & \\
T_nk_n\underline{X}_{n-1} & T_n^2k_n\underline{X}_{n-1} \arrow[l] & & & T_n^3k_n\underline{X}_{n-1} \arrow[dll, "\alpha"] \\
& & T_n^2k_n\underline{X}_{n-1}[\mu k_{n-1}] \arrow[ul, shift left] \arrow[ul, shift right] & &
\end{tikzcd}
$$

But $\bar{\alpha}$ is the pullback of α along $T_n^2\omega\underline{X}_n[T_n\omega\underline{X}_n]$ and thus the section $k_n\beta\underline{X}_{n-1}$ of α determines a section $\bar{\beta}\underline{X}_n$ of $\bar{\alpha}$, the monad $(T_{n-1}, \lambda_{n-1}, \mu_{n-1})$ being normalized. On the other hand, $\rho\underline{X}_n$ is split by a map $\gamma\underline{X}_n$ since T_n is $(\widehat{T_{n-1}})$, and the monad $(T_{n-1}, \lambda_{n-1}, \mu_{n-1})$ is equable. Consequently, $\beta\underline{X}_n = \gamma\underline{X}_n.\bar{\beta}\underline{X}_n$ is a section of $\alpha X_n = \bar{\alpha}.\rho X_n$. Axiom (1) is straightforward. Let us check axiom (2).

$$T_n^2\mu_n\underline{X}_n.\gamma T_n\underline{X}_n.\bar{\beta}T_n\underline{X}_n.(\gamma\underline{X}_n.\bar{\beta}.X_n)[1] = T_n^2\mu_n\underline{X}_n.\gamma_n T_n\underline{X}_n.\bar{\beta}T_n\underline{X}_n.\gamma X_n[1].\bar{\beta}X_n[1]. \qquad (*)$$

Let us denote by $\bar{\bar{\beta}}T_n\underline{X}_n$ the unique map $PX_n[p_0] \to P^2X_n$ such that $p_i.\bar{\bar{\beta}}T_nX_n = p_i$, $0 \leq i \leq 1$ and $\pi^2.\bar{\bar{\beta}}T_n\underline{X}_n = \beta T_{n-1}\underline{X}_{n-1}$. Then $P\gamma\underline{X}_n.\bar{\bar{\beta}}T_n\underline{X}_n = \bar{\beta}T_n\underline{X}_n.\gamma\underline{X}_n[1]$ and $(*) = T_n^2\mu_n\underline{X}_n.\gamma_n T_n\underline{X}_n.P\gamma X_n.\bar{\bar{\beta}}T_n\underline{X}_n.\bar{\beta}X_n[1] = \gamma\underline{X}_n.p_2.\bar{\bar{\beta}}T_n\underline{X}_n.\bar{\beta}\underline{X}_n[1] = \gamma\underline{X}_n.\bar{\beta}\underline{X}_n.p_2$ since $p_2.\beta T_{n-1}\underline{X}_{n-1}.\beta\underline{X}_{n-1}[1] = \beta\underline{X}_{n-1}.p_2$. $\square$

8. The last step

We are now going to prove that $Alg\,W_n$ is discretely cofibrant on the hypomorphisms.

The functor W_0 is the identity on $Pt\,\mathbb{B}$. So $Alg\,W_0$ is an identity functor on $Grd\,\mathbb{B} = K_1\mathbb{B}$ and satisfies the previous property.

We shall suppose from now on that $Alg\,W_k$ satisfies this property for all $k \leq n-1$. It implies obviously that the following square is a pullback:

$$
\begin{tikzcd}
Pt_{\tau_{n-2}}Alg\,\check{T}_{n-1} \arrow[r, "Pt_\tau\,Alg\,w_{n-1}"] \arrow[d, "p"'] & Pt_{\tau_{n-2}}Alg\,T_{n-1} \arrow[d, "p"] \\
Alg\,T_{n-1} \arrow[r, "Alg\,w_{n-1}"] & Alg\,T_{n-1}
\end{tikzcd}
$$

Proposition 8.1. *In presence of the previous inductive hypothesis the following square is a pullback*:

$$\begin{tikzcd}
\check{T}_n\mathbb{E} = \hat{T}_{n-1}(\check{T}_{n-1}\mathbb{E}) \arrow[r, "\hat{T}W_{n-1}"] \arrow[d, "\hat{\tau}_{n-1}"'] & \hat{T}_{n-1}(T_{n-1}\mathbb{E}) \arrow[d, "\hat{\tau}_{n-1}"] \\
\check{T}_{n-1}\mathbb{E} \arrow[r, "W_{n-1}"'] & T_{n-1}\mathbb{E}
\end{tikzcd}$$

Proof. Straightforward from the previous remark and Theorem 2.2. □

Theorem 8.2. *The functor Alg W_n is discretely cofibrant on the hypomorphisms.*

Proof. Let us recall that $W_n = TW_{n-1}.W_{\hat{T}} = W_T.\hat{T}W_{n-1}$. Let $(\underline{X}_n, x)$ be an algebra in $\check{T}_n\mathbb{E}$ and $f : W_n\underline{X}_n \to \underline{Y}_n$ a hypomorphism between the algebras $(W_n\underline{X}_n, W_nx)$ and $(\underline{Y}_n, y)$. Thus $\underline{Y}_{n-1} = \tau_{n-1}\underline{Y}_n = \tau_{n-1}W_n\underline{X}_n = W_{n-1}\underline{X}_{n-1}$. But $\omega W_n\underline{X}_n = W_n\omega\underline{X}_n$ is split by $W_n\sigma\underline{X}_n$ since X_n is in $\hat{T}_n\mathbb{E}$. So $f.W_n\sigma X_n = s$ is a splitting of $\omega\underline{Y}_n$.

The natural equation $x.T\sigma\underline{X}_n = x.\sigma T_n\underline{X}_n.k_n\tau_{n-1}\sigma\underline{X}_n = \sigma\underline{X}_n.k_n\tau_{n-1}x.k_n\tau_{n-1}\sigma\underline{X}_n$ in $\check{T}_n\mathbb{E}$, implies that $y.T_ns = s.k_n\tau_{n-1}(y.T_ns)$ in $T_n\mathbb{E} = T(T_{n-1}\mathbb{E})$. The monad $(T_{n-1}, \lambda_{n-1}, \mu_{n-1})$ being equable this section s determines a section γ_s of $\rho\underline{Y}_n : T_n^3\underline{X}_n \to P(X_n)$ (Theorem 6.2). On the other hand, we have $\underline{Y}_{n-1} = W_{n-1}\underline{X}_{n-1}$, and, the monad $(\widehat{T}_{n-1}, \hat{\lambda}_{n-1}, \hat{\mu}_{n-1})$ being normalized, we have a section $W_{n-1}\beta\underline{X}_{n-1}$ of $\alpha\underline{Y}_{n-1} : T_{n-1}^3\underline{Y}_{n-1} \to T_{n-1}^2\underline{Y}_{n-1}[\mu_{n-1}\underline{Y}_{n-1}]$.

We are now going to prove that the following left-hand side upper graph underlies a groupoid structure:

$$\begin{tikzcd}
T_n\underline{Y}_n \arrow[r] \arrow[ddd, "T_n\omega\underline{Y}_n"'] & T_n^2\underline{Y}_n \arrow[l, shift right=2, "T_ny"'] \arrow[l, shift left=2, "\mu_nY_n"] \arrow[ddd, "T_n^2\omega\underline{Y}_n"'] & & & T_n^3Y_{n-1} \arrow[lll, "T_n^2y"'] \arrow[dl, "\rho\underline{Y}_n"] \arrow[ddd, "T_n^3\omega\underline{Y}_n"] \\
 & & & P(\underline{Y}_n) \arrow[dl, "\bar{\alpha}"] \arrow[ddr, "\pi"] & \\
 & & T_n^2\underline{Y}_n[\mu_n\underline{Y}_n] \arrow[uul, shift left, "p_1"] \arrow[uul, shift right, "p_0"'] \arrow[dd] & & \\
T_nk_n\underline{Y}_{n-1} & T_n^2k_n\underline{Y}_{n-1} \arrow[l, "\mu_nk_nY_{n-1}"] & & & T_n^3k_nY_{n-1} \arrow[dll, "\alpha"] \\
 & & T_n^2k_n\underline{Y}_{n-1}[\mu_nk_nY_{n-1}] \arrow[ul, shift left, "p_1"] \arrow[ul, shift right, "p_0"'] & &
\end{tikzcd}$$

Indeed, the section $k_n(W_{n-1}\beta\underline{X}_{n-1})$ of α produces a section $\bar{\beta}$ of $\bar{\alpha}$. Then $\gamma_s.\bar{\beta}$ is a section of $\alpha\underline{Y}_n$ and thus $T_n^2y.\gamma_s.\bar{\beta}$ is a candidate to complete the groupoid structure since $\mu_n\underline{Y}_n.T_n^2y.\gamma_s.\bar{\beta} = T_ny.\mu_nT_n\underline{Y}_n.\gamma_s.\bar{\beta} = T_ny.p_0.\bar{\beta} = T_ny.p_0$ and $T_ny.T_n^2y.\gamma s.\bar{\beta} = T_ny.T_n\mu_n\underline{Y}_n.\gamma_s.\bar{\beta} = T_ny.p_1.\bar{\beta} = p_1$.

It remains to check the associativity axiom, that is $T_n^2y.\gamma_s.\bar{\beta}.\delta_3 = T_n^2y.\gamma_s.\bar{\beta}.p_2$ if δ_3 is the map defined by $p_i.\delta_3 = T_n^2y.\gamma_s.\bar{\beta}.p_i$, $0 \le i \le 1$. This map δ_3 is just $T_n^2y[T_ny].(\gamma_s.\bar{\beta})[1]$.

The whole result will follow from the following equality:

$$\gamma_s.\bar{\beta}.T_n^2 y[T_n y].\gamma_s[1].\bar{\beta}[1] = T^3 y.\gamma_s^2.\bar{\bar{\beta}}.\bar{\beta}[1],$$

where again $\bar{\bar{\beta}}$ is the factorization $P[p_0] \to P^2\underline{Y}_n$ determined by $W_{n-1}\beta T_{n-1}\underline{X}_{n-1}$: $T_{n-1}^2\underline{Y}_{n-1}[\mu_{n-1}\underline{Y}_{n-1}] \to T_{n-1}^3\underline{Y}_{n-1}$; it will be checked by composition with $T_n\mu_n\underline{Y}_n$ (straightforward) and with $T_n^2\omega T_n\underline{Y}_n$.

This last equality must be itself checked by composition with $T_{n-2}\omega T_{n-1}(\tau_{n-1}T_n\underline{Y}_n)$ because of the definition of γ_s by means of ψ_s. Finally, the equality holds in $T_{n-2}\mathbb{E}$ because of the equalities $y.T_n s = s.k_n\tau_{n-1}(y.T_n s)$ in $T_n\mathbb{E}$ and $\tau_{n-1}(y.T_n s) = \tau_{n-1}(W_n(x.T_n\sigma\underline{X}_n)) = W_{n-1}(\tau_{n-1}(x.T_n\sigma\underline{X}_n))$, the last one allowing the use of the naturality of β.

Then we have $T_n^2 y.\gamma_s.\bar{\beta}.T_n^2 y[T_n y].\gamma_s[1].\bar{\beta}[1] = T_n^2 y.T_n^3 y.\gamma^2 s.\bar{\bar{\beta}}.\bar{\beta}[1] = T_n^2 y.T_n^2\mu_n\underline{Y}_n.\gamma_s^2.\bar{\bar{\beta}}.\bar{\beta}[1] = T_n^2 y.\gamma_s.p_2.\bar{\bar{\beta}}.\bar{\beta}[1] = T_n^2 y.\gamma_s.\bar{\beta}.p_2$.

The groupoid structure implies that y is cartesian, consequently the section s of $\omega\underline{Y}_n$ can be pulled back to a section σ of $\omega T_n\underline{Y}_n$. It is easy to check that (Y_n,σ) in an object of $\hat{T}_{n-1}(T_{n-1}\mathbb{E})$ in the same way as the map f is a map in this category. But we have $\hat{\tau}_{n-1}(\underline{Y}_n,\sigma) = \underline{Y}_{n-1} = W_{n-1}\underline{X}_{n-1}$. Thus according to the previous proposition, it determines a unique $\bar{\bar{Y}}_n$ in $\hat{T}_n\mathbb{E}$ such that $W_n\bar{\underline{Y}}_n = (Y_n,\sigma)$ and $\tau_{n-1}\bar{\bar{Y}}_n = \underline{X}_{n-1}$. The equality $\hat{\tau}_{n-1}(f) = W_{n-1}(1_{\underline{X}_{n-1}})$ determines a unique $\bar{f} = X_n \to \bar{\bar{Y}}_n$ above f.

We have then completed the proof that $Alg\, W_n$, the nerve functor, is discretely cofibrant on the hypomorphisms. Consequently, we have characterized the groupoids among the simplicial objects. □

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